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Elasticity, seismic events and microseismic monitoring

By "seismic events" we understand earthquakes of any size. There exists a broad scientific literature on earthquakes and on the processing of seismologic data. We refer readers interested in a detailed description of these subjects to corresponding books (see, for example, Lay and Wallace, 1995, and Shearer, 2009). We start this book with an introductory review of the theory of linear elasticity and of the mechanics of seismic events. The aim of this chapter is to describe classical fundamentals of the working frame necessary for our consideration of induced seismicity. We conclude this chapter with a short introduction to methodical aspects of the microseismic monitoring.

1.1 Linear elasticity and seismic waves

Deformations of a solid body are motions under which its shape and (or) its size change. Formally, deformations can be described by a field of a displacement vector $\mathbf{u}(\mathbf{r})$. This vector is a function of a location $\mathbf{r}$ of any point of the body in an initial reference state (e.g., the so-called unstrained configuration; see, for example, Segall, 2010). Initially we accept here the so-called Lagrangian formulation, i.e. we observe motions of a given particle of the body.

However, the field of displacements describes not only deformations of the body but also its possible rigid motions without changes of its shape and its size, such as translations and/or rotations.

In contrast to rigid motions, under deformations, distances (some or any) between particles of the body change. Therefore, to describe deformations, a mathematical function of the displacement field is used that excludes rigid motions of a solid and describes changes of distances between its particles only. This function is the strain tensor $\boldsymbol{\epsilon}$, which is a second-rank tensor with nine components ϵ_{ij}. Here the indices i and j can accept any of values 1, 2 and 3 denoting the coordinate directions of a Cartesian coordinate system in which the vectors $\mathbf{u}$ and $\mathbf{r}$ have been defined.

1.1.1 Strain

In the case of small deformations (i.e. where absolute values of all spatial derivatives of any components of the vector $\mathbf{u}(\mathbf{r})$ are much smaller than 1) the strain tensor has the form of a 3×3 symmetric matrix with the following components:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.1}$$

This form of the strain tensor describes deformations within a small vicinity of a given location. This form remains the same also by consideration of small deformations in the Eulerian formulation (see Segall, 2010), where instead of a motion of a given particle of the body (i.e. the Lagrangian approach) rather a motion at a given coordinate location (i.e. at a given point of the space) is considered. In this book we accept the small-deformation approximation and do not distinguish between the Lagrangian and Eulerian approaches.

Strains ϵ_{ij} can be arbitrary (small) numbers. However, because of their definition (1.1) they cannot be arbitrarily distributed in space. Spatial derivatives of strains must be constrained by the following compatibility equations (see Segall, 2010):

$$\frac{\partial^2 \epsilon_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 \epsilon_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 \epsilon_{ik}}{\partial x_j \partial x_l} + \frac{\partial^2 \epsilon_{jl}}{\partial x_k \partial x_i}. \tag{1.2}$$

Deformations of a body results from applications of loads to it. Deformations that will disappear completely if the loads are released are called elastic. Bodies that can have elastic deformations are called elastic bodies.

1.1.2 Stress

Elastic bodies resist their elastic deformations by means of elastic forces. Elastic forces in a solid body are analogous to a pressure in an ideal fluid. They occur due to mutual interactions of elastically deformed parts of the body. These interactions in turn take place on surfaces where the parts of the body are contacting each other (see also Landau and Lifshitz, 1987).

Let us consider an elementary part of a body under deformation (see Figure 1.1). Other parts of the body act by means of elastic forces onto this elementary part over its surface S. Let us consider a differentially small element of this surface at its arbitrary point $\mathbf{r}$. Such a surface element can be approximated by a differentially small part of a plane of area dS tangential to S at point $\mathbf{r}$ with a unit normal $\mathbf{n}$ directed outside this part of the surface. Owing to elastic deformations an elastic force $d\mathbf{F}(\mathbf{r}, \mathbf{n})$ (also called a stress force) acts on the plane element with the normal $\mathbf{n}$. The following limit defines a traction vector:

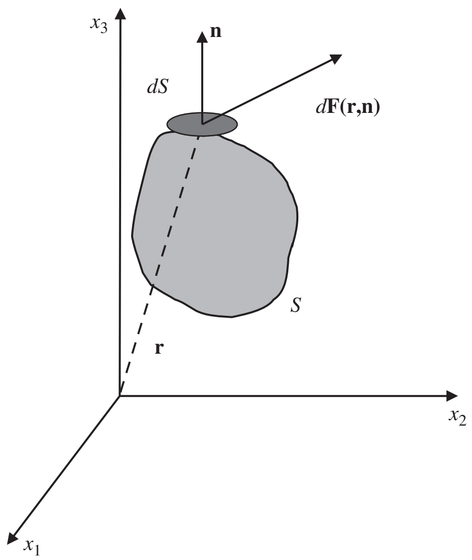

Figure 1.1 A sketch for defining a traction.

$$\boldsymbol{\tau}(\mathbf{r}, \mathbf{n}) = \lim_{dS \to 0} \frac{d\mathbf{F}(\mathbf{r}, \mathbf{n})}{dS}. \tag{1.3}$$

Note that the traction has the same physical units as a pressure in a fluid (e.g. Pa in the SI system). Note also that the traction is a function of a location **r** and of an orientation of the normal **n**.

Let us consider three plane elements parallel to coordinate planes at a given location. We assume also that their normals point in the positive directions of coordinate axes, which are perpendicular to the plane elements. Therefore, the corresponding three normals coincide with the unit basis vectors $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$ and $\hat{\mathbf{x}}_3$ of the Cartesian coordinate system under consideration. Tractions acting on these plane elements are $\boldsymbol{\tau}(\mathbf{r}, \hat{\mathbf{x}}_1)$, $\boldsymbol{\tau}(\mathbf{r}, \hat{\mathbf{x}}_2)$ and $\boldsymbol{\tau}(\mathbf{r}, \hat{\mathbf{x}}_3)$, respectively. A 3×3 matrix composed of nine coordinate components of these tractions defines the stress tensor, $\boldsymbol{\sigma}$. Its element σ_{ij} denotes the ith component of the traction acting on the surface with the normal $\hat{\mathbf{x}}_j$:

$$\sigma_{ij} = \tau_i(\hat{x}_j). \tag{1.4}$$

Let us consider a differentially small elastic body under an elastic strain and assume for all deformation processes enough time to bring parts of this body into an equilibrium state. From the equilibrium conditions for the rotational moments (torques) of elastic forces it follows that the stress tensor is symmetric:

$$\sigma_{ij} = \sigma_{ji}. \tag{1.5}$$

Note that if the body torques are negligible (which is usually the case) this relation is valid even in the case of the presence of rotational motions. This is because of the fact that, in the limit of a small elementary volume, the inertial forces are decreasing faster than the elastic force torques (see Auld, 1990, volume 1, section 2, for more details).

Similarly, a consideration of forces (elastic forces, body forces and inertial forces) acting on a volume element in the limit of its vanishing volume shows that elastic forces applied to the surface of such a volume must be in balance (see Auld, 1990, volume 1, section 2, for more details). It then follows that a traction $\boldsymbol{\tau}(\mathbf{r}, \mathbf{n})$ acting on an arbitrarily oriented plane surface element can be computed by using the stress tensor:

$$\tau_i(\mathbf{n}) = \sigma_{ij} n_j. \tag{1.6}$$

Note that here and generally in this book (if not specially mentioned) we accept the agreement on summation on repeated indices, e.g. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$.

Definition (1.4) of the stress tensor corresponds to a common continuum mechanics sign convention that tensile stresses are positive and compressive stresses are negative (see, for example, a thin elementary volume and tractions acting on its outer surface with normals pointing outside this volume; Figure 1.2).

1.1.3 Stress–strain relations

The strain-tensor and stress-tensor notations give a general form of an observational fact, known as Hooke's law, that small elastic deformations are proportional to elastic forces:

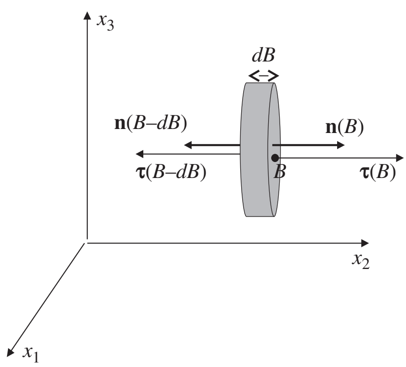

Figure 1.2 A sketch illustrating positiveness of tensile stresses. Indeed, equation (1.6) requires that the components σ_{22} in the both points, B and $B - dB$ must be positive. Note that the point B is shown as a dot on the right-hand side of the disc. The point denoted as $B - dB$ is not seen. It is on the left-hand side of the disc; dB denotes the width of the disc.

$$\epsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{1.7}$$

where the fourth-rank tensor **S**, with components S_{ijkl}, is the tensor of elastic compliances. Note that their physical units are inverse to the unit of stress: 1/Pa. Owing to the symmetry of the strain and stress tensors, the tensor of elastic compliances has the following symmetries:

$$S_{ijkl} = S_{jikl} = S_{ijlk}. \tag{1.8}$$

Another fourth-rank tensor **C**, with components C_{ijkl}, called the tensor of elastic stiffnesses, yields an alternative formulation of Hooke's law:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}. \tag{1.9}$$

From this equation it is clear that the tensor of elastic stiffnesses also has the symmetry:

$$C_{ijkl} = C_{jikl} = C_{ijlk}. \tag{1.10}$$

Both the tensor of elastic stiffnesses and the tensor of elastic compliances are physical characteristics of a given elastic body.

Often both forms of Hooke's law (1.7) and (1.9) are written symbolically as (see Auld, 1990):

$$\boldsymbol{\epsilon} = \mathbf{S} : \boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon}. \tag{1.11}$$

Here the double-dot (or double scalar) products denote summations over pairs of repeating indices in (1.7) and (1.9), respectively.

A deformed elastic body possesses an elastic strain energy. At zero strain this energy is equal to zero. With increasing strain by an increment $d\epsilon_{kl}$ due to the stress σ_{kl}, the volumetric density of this energy (energy per unit volume) must increase by the increment $dE = \sigma_{kl}d\epsilon_{kl}$ (see Landau and Lifshitz, 1987). The tensor of elastic stiffnesses can then be used to define the density of the elastic strain energy (by integration of the increment dE) as a positive quadratic function of non-zero strains:

$$E = \frac{1}{2}C_{ijkl}\epsilon_{ij}\epsilon_{kl} = \frac{1}{2}\sigma_{kl}\epsilon_{kl} = \frac{1}{2}S_{klij}\sigma_{kl}\sigma_{ij}, \tag{1.12}$$

where in the two last expressions the two forms of Hooke's law (1.7) and (1.9) have been used. The product $\epsilon_{ij}\epsilon_{kl}$ remains unchanged if the index pair ij is replaced by kl and kl is replaced by ij, respectively. Thus, the tensor of elastic stiffnesses as well as the the tensor of compliances must also have the following symmetry:

$$C_{ijkl} = C_{klij}, \quad S_{ijkl} = S_{klij}. \tag{1.13}$$

Symmetries (1.8), (1.10) and (1.13) of the stiffness and compliance tensors reduce the number of their independent components. From 81 possible components of a tensor (tensors' indices can be equal to 1, 2 or 3) only 21 components are mutually independent. These components are also called elastic moduli or elastic constants (the latter notation neglects such effects as pressure dependence and temperature dependence of these quantities). The requirement that the elastic strain energy must be a positive-definite quadratic form of arbitrary strain/stress components (called also the stability condition) provides additional restrictions on allowed values of elastic moduli.

1.1.4 Elastic moduli

The tensors C_{ijkl} and S_{ijkl} are inverse to each other so that (see Cheng, 1997):

$$C_{ijkl}S_{klmn} = \frac{1}{2}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}), \tag{1.14}$$

where quantity δ_{kl} is the so-called Kronecker matrix, with components $\delta_{kl} = 1$, for $k = l$, and $\delta_{ij} = 0$ in other cases.

The tensors of stiffnesses and compliances can be expressed in convenient matrix forms by using their 21 independent components, respectively. For this, one uses the so-called contracted notation (or the Voigt notations). Let us introduce capital indices (e.g. I, J, etc.), which can take values 1, 2, 3, 4, 5 and 6. The following correspondence between the capital indices and the pairs of the usual indices (ij) is assigned: $1 \to 11$, $2 \to 22$, $3 \to 33$, $4 \to 23$, $5 \to 13$, and $6 \to 12$. In these notations Hooke's law has the following forms (Jaeger *et al.*, 2007; Auld, 1990):

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{12} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{13} & s_{23} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{14} & s_{24} & s_{34} & s_{44} & s_{45} & s_{46} \\ s_{15} & s_{25} & s_{35} & s_{45} & s_{55} & s_{56} \\ s_{16} & s_{26} & s_{36} & s_{46} & s_{56} & s_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix}, \tag{1.15}$$

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}. \tag{1.16}$$

In these two equations the contracted notation is used in the two symmetric 6×6 matrices of components s_{IK} and c_{IK}, where I corresponds to a pair of normal indices, e.g. ij, and K corresponds to another their pair, e.g. kl. It is clear that these matrices are inverse to each other, i.e. their matrix product gives a 6×6 unit matrix. Their components are also called elastic compliances and elastic stiffnesses, respectively. The relations between the contracted-notation stiffness matrix components and corresponding components of the fourth-rank tensor of elastic stiffnesses is simple: $c_{IK} = C_{ijkl}$. This correspondence for the compliances is a bit more complicated: $s_{IK} = S_{ijkl}$ if $I, K = 1, 2, 3$, $s_{IK} = 2S_{ijkl}$ if $I = 1, 2, 3$ and $K = 4, 5, 6$, and $s_{IK} = 4S_{ijkl}$ if $I, K = 4, 5, 6$.

The higher the physical symmetry of the elastic medium, the smaller is the number of non-vanishing independent elastic constants. For mineral crystals, different symmetries are of relevance (see Auld, 1990, for a comprehensive description). In the most general case of triclinic crystals the elastic properties are characterized by 21 independent compliances (or, equivalently, 21 independent stiffnesses). This situation corresponds to equations (1.15) and (1.16), respectively. If the medium has a single symmetry plane (the monoclinic symmetry) then the number of independent constants will be reduced to 13 (for example, if we assume the xy coordinate plane as the plane of symmetry, this will result in the invariant coordinate transformation $z \to -z$ and thus, all elastic constants with odd numbers of index 3 must be equal to zero). This situation corresponds, for example, to a layered medium with a single system of plane cracks oblique to the lamination plane.

One of most relevant symmetries for rocks is the orthorhombic one. It can be applied to describe different geological situations, like rocks with three mutually perpendicular systems of cracks or horizontally layered rocks permeated by a single system of aligned vertical fractures. An orthorhombic medium has three mutually perpendicular symmetry planes. This means that in such a medium under corresponding coordinate transformations (reflections across symmetry planes) the tensors of elastic constants must remain unchanged. In a coordinate system with axes normal to the symmetry planes it follows that all components C_{ijkl} and S_{ijkl} with odd numbers of any index must be equal to zero. This leads to the following forms of the compliance and stiffness matrices, respectively:

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22} & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66} \end{bmatrix}; \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}. \quad (1.17)$$

We see that nine independent constants are enough to completely describe the elastic properties of an orthorhombic medium. The compliances can be obtained from stiffnesses by the matrix inversion and vice versa. In the case of an arbitrary coordinate orientation, three additional constants (corresponding to three rotational angles) are required.

A useful and geologically relevant subset of orthorhombic symmetry is transverse isotropy. Layered sedimentary rocks can frequently be described by this symmetry. The plane of lamination is then the symmetry plane. If one of the coordinate planes coincides with the symmetry plane, then a coordinate axis normal to the symmetry plane will be an axis of an arbitrary-angle rotational symmetry. This symmetry results in four additional relations between the elastic constants, reducing the number of independent ones to five. If the symmetry axis coincides with the direction of the axis x_3, then in equations (1.17) additional relations will be (Auld, 1990): $s_{22} = s_{11}$, $s_{23} = s_{13}$, $s_{55} = s_{44}$ and $s_{66} = 2(s_{11} - s_{12})$. Correspondingly, $c_{22} = c_{11}$, $c_{23} = c_{13}$, $c_{55} = c_{44}$ and $c_{66} = (c_{11} - c_{12})/2$.

Finally, in the case of an elastic isotropic medium (all coordinate axes are arbitrary-angle rotational symmetry axes and any plane is a plane of symmetry), two constants remain independent only: $s_{22} = s_{33} = s_{11}$, $s_{23} = s_{13} = s_{12}$, $s_{66} = s_{55} = s_{44}$ and $s_{44} = 2(s_{11} - s_{12})$. Correspondingly, $c_{22} = c_{33} = c_{11}$, $c_{23} = c_{13} = c_{12}$, $c_{66} = c_{55} = c_{44}$ and $c_{44} = (c_{11} - c_{12})/2$. The independent elastic stiffnesses are usually denoted as the elastic moduli λ and μ, so that $c_{44} = \mu$ and $c_{12} = \lambda$. Inverting the matrix c_{ij} we obtain compliances of an isotropic medium:

$$s_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}, \quad s_{12} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}, \quad s_{44} = \frac{1}{\mu}. \tag{1.18}$$

Let us consider a volumetric strain (dilatation) of an elementary volume V of an arbitrary anisotropic elastic medium:

$$\epsilon \equiv \frac{dV}{V}. \tag{1.19}$$

We can choose such an elementary volume to be a cuboid with side lengths l_x, l_y and l_z. Thus we see that

$$\epsilon = \frac{d(l_x l_y l_z)}{l_x l_y l_z} = \frac{dl_x}{l_x} + \frac{dl_y}{l_y} + \frac{dl_z}{l_z} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}. \tag{1.20}$$

Let us further assume that this dilatation is a result of a hydrostatic stress, $\sigma_{kl} = -p\delta_{kl}$, applied to the medium, where p is the pressure loading the medium. A general relation between the dilatation and the pressure can be obtained by taking a double-dot product (the scalar product) of Hooke's law (1.7) with the δ_{ij} (i.e. multiplying the both sides with δ_{ij} and summing up over repeating indices):

$$\epsilon = -S_{iikk}\, p. \tag{1.21}$$

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The proportionality coefficient here is a bulk compressibility C^{mt} of the elastic material:

$$C^{mt} \equiv S_{iikk} = S_{1111} + S_{2222} + S_{3333} + 2(S_{1122} + S_{1133} + S_{2233})$$
$$= s_{11} + s_{22} + s_{33} + 2(s_{12} + s_{13} + s_{23}). \quad (1.22)$$

It follows from (1.19)–(1.21) that the bulk compressibility of a sample has the following relation to its bulk density ρ:

$$C^{mt} = -\frac{dV}{V dp} = -\frac{d(1/\rho)}{(1/\rho)dp} = \frac{1}{\rho}\frac{d\rho}{dp}. \quad (1.23)$$

In the case of an isotropic elastic material we obtain (see equations (1.22) and (1.18)) $C^{mt} = 3s_{11} + 6s_{12} = 1/(\lambda + 2\mu/3)$. Therefore,

$$K = \lambda + 2\mu/3 \quad (1.24)$$

is a bulk modulus describing the stiffness of the material to volumetric deformations.

The following representation of the stiffness tensor of an isotropic medium is useful (Aki and Richards, 2002):

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (1.25)$$

In the same terms, Hooke's law for isotropic elastic media can be written in the following form:

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon + 2\mu\epsilon_{ij}. \quad (1.26)$$

From this equation it follows that μ is a shear modulus of the material, describing its stiffness to shear deformations (under which $i \neq j$). It follows also that under uniaxial stress conditions (for example $\sigma_{33} \neq 0$ and $\sigma_{11} = \sigma_{22} = 0$) the ratio ν of the transverse strain to the longitudinal strain, $-\epsilon_{11}/\epsilon_{33}$, is equal to

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \quad (1.27)$$

This quantity is called Poisson's ratio. For an isotropic elastic solid the stability condition requires that both bulk and shear moduli must be positive. For Poisson's ratio this yields the restriction $-1 \leq \nu \leq 0.5$. For realistic rocks this coefficient is positive. Its upper limit of 0.5 corresponds to fluids. Frequently, its values for stiff tight isotropic rocks are close to 0.25 (corresponding to $\lambda \approx \mu$).

All elastic moduli introduced above will usually be assumed to be isothermal ones, if static deformations or processes being very slow in respect to the thermal diffusion are considered. In this book we consider processes that are faster than the temperature equilibration (e.g. wave propagation and pore-pressure equilibration). We will assume that these processes are approximately adiabatic. Thus, we assume that the elastic moduli introduced above are adiabatic. Note that the adiabatic and isothermal moduli of hard materials (e.g. rocks) differ by a small amount (see also Landau and Lifshitz, 1987).

In this book we will frequently assume that the elastic properties of the medium are isotropic. This simplifying assumption is often too rough for problems of seismic event location and imaging (which are not the main subject of our consideration). For such problems velocity models should take into account seismic anisotropy at least in the weak anisotropy approximation (Thomsen, 1986; Tsvankin, 2005; Grechka, 2009). For describing dominant effects responsible for the triggering of induced microseismicity the assumption of elastic isotropy seems to be adequate at least as the first approximation. For such effects hydraulic anisotropy of rocks is much more important. Elastic anisotropy in rocks is usually below 10% and seldom exceeds 30%, in respect to the velocity contrast between the slowest and fastest wave propagation directions. In shale the elastic anisotropy can be even higher. However, usually it is much smaller than a possible anisotropy of the hydraulic permeability, which can reach several orders of magnitude.

1.1.5 Dynamic equations and elastic waves

By an elastic deformation, a transfer of an elastic solid from one equilibrium state to another equilibrium state occurs by means of propagation of elastic waves. Elastic waves in rocks in the frequency range between 10^{-3} and 10^4 Hz are usually referred to as seismic waves. Resulting elastic forces acting on an elementary volume of the elastic medium define its acceleration vector. Owing to Hooke's law and the definition of the strain tensor, the second Newtonian law (i.e. the momentum conservation) takes the form of the following dynamic equation (Lamé equation):

$$\frac{\partial}{\partial x_j} C_{ijkl} \frac{\partial u_k}{\partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2}. \qquad (1.28)$$

This equation describes the propagation of elastic waves in the most general case of a heterogeneous anisotropic elastic medium. Note that this is a system of three equations for three unknown components of the displacement vector. A plane-wave analysis (see also our later discussion of poroelastic waves) is instructive for investigating modes of propagation of elastic perturbations.

Let us consider the case of a homogeneous arbitrary anisotropic elastic medium. Then equation (1.28) simplifies to:

$$C_{ijkl} \frac{\partial^2 u_k}{\partial x_j x_l} = \rho \frac{\partial^2 u_i}{\partial t^2}. \qquad (1.29)$$