## Contents

**Preface** ........................................... page xv

**Introduction** ................................. 1

1 Nonlinear Theories of Elasticity of Plates and Shells .............. 6

1.1 Introduction ................................. 6

1.1.1 Literature Review ......................... 6

1.2 Large Deflection of Rectangular Plates ......................... 8

1.2.1 Green's and Almansi Strain Tensors for Finite Deformation 8

1.2.2 Strains for Finite Deflection of Rectangular Plates: Von Kármán Theory 11

1.2.3 Geometric Imperfections .................... 14

1.2.4 Eulerian, Lagrangian and Kirchhoff Stress Tensors ............ 14

1.2.5 Equations of Motion in Lagrangian Description ............... 18

1.2.6 Elastic Strain Energy ....................... 18

1.2.7 Von Kármán Equation of Motion ................. 19

1.2.8 Von Kármán Equation of Motion Including Geometric Imperfections 23

1.3 Large Deflection of Circular Cylindrical Shells ................. 24

1.3.1 Euclidean Metric Tensor ..................... 24

1.3.1.1 Example: Cylindrical Coordinates ........... 25

1.3.1.2 Example: Spherical Coordinates .............. 25

1.3.2 Green's Strain Tensor in a Generic Coordinate System ....... 26

1.3.3 Green's Strain Tensor in Cylindrical Coordinates ............. 27

1.3.4 Strains for Finite Deflection of Circular Cylindrical Shells: Donnell's Nonlinear Theory 29

1.3.5 Geometric Imperfections in Donnell's Nonlinear Shell Theory .... 32

1.3.6 The Flügge-Lur’e-Byrne Nonlinear Shell Theory ............ 32

1.3.7 The Novozhilov Nonlinear Shell Theory .................. 34

1.3.8 The Sanders-Koiter Nonlinear Shell Theory ................. 36

1.3.9 Elastic Strain Energy .......................... 36

1.3.10 Donnell's Nonlinear Shallow-Shell Theory ................. 37
## 1.3.11 Donnell’s Nonlinear Shallow-Shell Theory Including Geometric Imperfections 43

### 1.4 Large Deflection of Circular Plates

- **1.4.1 Green’s Strain Tensor for Circular Plates** 43
- **1.4.2 Strains for Finite Deflection of Circular Plates: Von Kármán Theory** 44
- **1.4.3 Von Kármán Equation of Motion for Circular Plates** 45

### 1.5 Large Deflection of Spherical Caps

- **1.5.1 Green’s Strain Tensor in Spherical Coordinates** 47
- **1.5.2 Strains for Finite Deflection of Spherical Caps: Donnell’s Nonlinear Theory** 47
- **1.5.3 Donnell’s Equation of Motion for Shallow Spherical Caps** 48
- **1.5.4 The Flügge-Lur’e-Byrne Nonlinear Shell Theory** 49

## 2 Nonlinear Theories of Doubly Curved Shells for Conventional and Advanced Materials ........................................... 52

### 2.1 Introduction 52

### 2.2 Doubly Curved Shells of Constant Curvature

- **2.2.1 Elastic Strain Energy** 55

### 2.3 General Theory of Doubly Curved Shells

- **2.3.1 Theory of Surfaces** 56
- **2.3.2 Green’s Strain Tensor for a Shell in Curvilinear Coordinates** 62
- **2.3.3 Strain-Displacement Relationships for Novozhilov’s Nonlinear Shell Theory** 65
- **2.3.4 Strain-Displacement Relationships for an Improved Version of the Novozhilov Shell Theory** 67
- **2.3.5 Simplified Strain-Displacement Relationships** 68
- **2.3.6 Elastic Strain Energy** 69
- **2.3.7 Kinetic Energy** 69

### 2.4 Composite and Functionally Graded Materials

- **2.4.1 Stress-Strain Relations for a Thin Lamina** 71
- **2.4.2 Stress-Strain Relations for a Layer within a Laminate** 73
- **2.4.3 Elastic Strain Energy for Laminated Shells** 73
- **2.4.4 Elastic Strain Energy for Orthotropic and Cross-Ply Shells** 74
- **2.4.5 Sandwich Plates and Shells** 75
- **2.4.6 Functionally Graded Materials and Thermal Effects** 76

### 2.5 Nonlinear Shear Deformation Theories for Moderately Thick, Laminated and Functionally Graded, Doubly Curved Shells

- **2.5.1 Nonlinear First-Order Shear Deformation Theory for Doubly Curved Shells of Constant Curvature** 78
- **2.5.2 Elastic Strain Energy for Laminated Shells** 80
- **2.5.3 Kinetic Energy with Rotary Inertia for Laminated Shells** 81
- **2.5.4 Nonlinear Higher-Order Shear Deformation Theory for Laminated, Doubly Curved Shells** 81
- **2.5.5 Elastic Strain and Kinetic Energies, Including Rotary Inertia, for Laminated Shells According with Higher-Order Shear Deformation Theory** 85
- **2.5.6 Elastic Strain Energy for Heated, Functionally Graded Shells** 86
Contents

2.5.7 Kinetic Energy with Rotary Inertia for Functionally Graded Shells 87
2.6 Thermal Effects on Plates and Shells 88
References 89

3 Introduction to Nonlinear Dynamics .......................... 90
3.1 Introduction 90
3.2 Periodic Nonlinear Vibrations: Softening and Hardening Systems 90
3.3 Numerical Integration of the Equations of Motion 93
3.4 Local Geometric Theory 94
3.5 Bifurcations of Equilibrium 96
3.5.1 Saddle-Node Bifurcation 97
3.5.2 Pitchfork Bifurcation 97
3.5.3 Transcritical Bifurcation 99
3.5.4 Hopf Bifurcation 99
3.6 Poincaré Maps 101
3.7 Bifurcations of Periodic Solutions 103
3.7.1 Floquet Theory 103
3.7.2 Period-Doubling Bifurcation 106
3.7.3 Neimark-Sacker Bifurcation 107
3.8 Numerical Continuation Methods 107
3.8.1 Arclength Continuation of Fixed Points 107
3.8.2 Pseudo-Arclength Continuation of Fixed Points 109
3.8.3 Pseudo-Arclength Continuation of Periodic Solutions 110
3.9 Nonlinear and Internal Resonances 112
3.10 Chaotic Vibrations 113
3.11 Lyapunov Exponents 114
3.11.1 Maximum Lyapunov Exponent 114
3.11.2 Lyapunov Spectrum 115
3.12 Lyapunov Dimension 117
3.13 Discretization of the System: Galerkin Method and Lagrange Equations 118
References 119

4 Vibrations of Rectangular Plates ............................... 120
4.1 Introduction 120
4.1.1 Literature Review 120
4.2 Linear Vibrations with Classical Plate Theory 121
4.2.1 Theoretical and Experimental Results 122
4.3 Nonlinear Vibrations with Von Kármán Plate Theory 123
4.3.1 Boundary Conditions, Kinetic Energy, External Loads and Mode Expansion 124
4.3.2 Satisfaction of Boundary Conditions 127
4.3.2.1 Case (a) 127
4.3.2.2 Case (b) 128
4.3.2.3 Case (c) 128
4.3.2.4 Case (d) 129
4.3.3 Lagrange Equations of Motion 130
4.4 Numerical Results for Nonlinear Vibrations 131
4.5 Comparison of Numerical and Experimental Results 132
# Table of Contents

4.6 Inertial Coupling in the Equations of Motion 137
4.7 Effect of Added Masses 139
References 139

5 Vibrations of Empty and Fluid-Filled Circular Cylindrical Shells 141
5.1 Introduction 141
  5.1.1 Literature Review 142
5.2 Linear Vibrations of Simply Supported, Circular Cylindrical Shells 145
  5.2.1 Donnell’s Theory of Shells 145
  5.2.2 Flügge-Lur’e-Byrne Theory of Shells 148
5.3 Circular Cylindrical Shells Containing or Immersed in Still Fluid 150
5.4 Rayleigh-Ritz Method for Linear Vibrations 154
5.5 Nonlinear Vibrations of Empty and Fluid-Filled, Simply Supported, Circular Cylindrical Shells with Donnell’s Nonlinear Shallow-Shell Theory 156
  5.5.1 Fluid-Structure Interaction 159
  5.5.2 Stress Function and Galerkin Method 160
  5.5.3 Traveling-Wave Response 164
  5.5.4 Proof of the Continuity of the Circumferential Displacement 164
5.6 Numerical Results for Nonlinear Vibrations of Simply Supported Shells 165
  5.6.1 Empty Shell 165
  5.6.2 Water-Filled Shell 171
5.7 Effect of Geometric Imperfections 171
  5.7.1 Empty Shell 172
  5.7.2 Water-Filled Shell 174
5.8 Comparison of Numerical and Experimental Results 176
  5.8.1 Empty Shell 180
  5.8.2 Water-Filled Shell 182
5.9 Chaotic Vibrations of a Water-Filled Shell 187
References 191

6 Reduced-Order Models: Proper Orthogonal Decomposition and Nonlinear Normal Modes 193
6.1 Introduction 193
6.2 Reference Solution 194
6.3 Proper Orthogonal Decomposition (POD) Method 194
6.4 Asymptotic Nonlinear Normal Modes (NNMs) Method 197
6.5 Discussion on POD and NNMs 199
6.6 Numerical Results 201
  6.6.1 Results for POD and NNMs Methods 202
  6.6.2 Geometrical Interpretation 207
References 209

7 Comparison of Different Shell Theories for Nonlinear Vibrations and Stability of Circular Cylindrical Shells 212
7.1 Introduction 212
7.2 Energy Approach 212
  7.2.1 Additional Terms to Satisfy the Boundary Conditions 215
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.2.2 Fluid-Structure Interaction</td>
<td>216</td>
</tr>
<tr>
<td>7.2.3 Lagrange Equations of Motion</td>
<td>217</td>
</tr>
<tr>
<td>7.3 Numerical Results for Nonlinear Vibrations</td>
<td>219</td>
</tr>
<tr>
<td>7.3.1 Empty Shell</td>
<td>219</td>
</tr>
<tr>
<td>7.3.1.1 Comparison with Results Available in the Literature</td>
<td>223</td>
</tr>
<tr>
<td>7.3.2 Water-Filled Shell</td>
<td>224</td>
</tr>
<tr>
<td>7.3.2.1 Water-Filled Shell with Imperfections</td>
<td>228</td>
</tr>
<tr>
<td>7.3.3 Discussion</td>
<td>230</td>
</tr>
<tr>
<td>7.4 Effect of Axial Load and Pressure on the Nonlinear Stability and Response of the Empty Shell</td>
<td>230</td>
</tr>
<tr>
<td>References</td>
<td>233</td>
</tr>
<tr>
<td>8 Effect of Boundary Conditions on Large-Amplitude Vibrations of Circular Cylindrical Shells</td>
<td>234</td>
</tr>
<tr>
<td>8.1 Introduction</td>
<td>234</td>
</tr>
<tr>
<td>8.1.1 Literature Review</td>
<td>234</td>
</tr>
<tr>
<td>8.2 Theory</td>
<td>235</td>
</tr>
<tr>
<td>8.3 Numerical Results</td>
<td>237</td>
</tr>
<tr>
<td>8.3.1 Comparison with Numerical and Experimental Results Available for Empty Shells</td>
<td>239</td>
</tr>
<tr>
<td>References</td>
<td>240</td>
</tr>
<tr>
<td>9 Vibrations of Circular Cylindrical Panels with Different Boundary Conditions</td>
<td>242</td>
</tr>
<tr>
<td>9.1 Introduction</td>
<td>242</td>
</tr>
<tr>
<td>9.1.1 Literature Review</td>
<td>242</td>
</tr>
<tr>
<td>9.2 Linear Vibrations</td>
<td>244</td>
</tr>
<tr>
<td>9.3 Nonlinear Vibrations</td>
<td>245</td>
</tr>
<tr>
<td>9.3.1 Mode Expansion</td>
<td>247</td>
</tr>
<tr>
<td>9.3.2 Satisfaction of Boundary Conditions</td>
<td>248</td>
</tr>
<tr>
<td>9.3.2.1 Model A</td>
<td>248</td>
</tr>
<tr>
<td>9.3.2.2 Model B</td>
<td>250</td>
</tr>
<tr>
<td>9.3.2.3 Model C</td>
<td>251</td>
</tr>
<tr>
<td>9.3.3 Solution</td>
<td>251</td>
</tr>
<tr>
<td>9.4 Numerical Results</td>
<td>252</td>
</tr>
<tr>
<td>9.4.1 Nonperiodic Response</td>
<td>256</td>
</tr>
<tr>
<td>9.5 Comparison of Experimental and Numerical Results</td>
<td>262</td>
</tr>
<tr>
<td>9.5.1 Experimental Results</td>
<td>262</td>
</tr>
<tr>
<td>9.5.2 Comparison of Numerical and Experimental Results</td>
<td>265</td>
</tr>
<tr>
<td>References</td>
<td>270</td>
</tr>
<tr>
<td>10 Nonlinear Vibrations and Stability of Doubly Curved Shallow-Shells: Isotropic and Laminated Materials</td>
<td>272</td>
</tr>
<tr>
<td>10.1 Introduction</td>
<td>272</td>
</tr>
<tr>
<td>10.1.1 Literature Review</td>
<td>272</td>
</tr>
<tr>
<td>10.2 Theoretical Approach for Simply Supported, Isotropic Shells</td>
<td>274</td>
</tr>
<tr>
<td>10.2.1 Boundary Conditions</td>
<td>275</td>
</tr>
<tr>
<td>10.2.2 Lagrange Equations of Motion</td>
<td>277</td>
</tr>
<tr>
<td>10.3 Numerical Results for Simply Supported, Isotropic Shells</td>
<td>279</td>
</tr>
<tr>
<td>10.3.1 Case with $R_x/R_y = 1$, Spherical Shell</td>
<td>279</td>
</tr>
</tbody>
</table>
10.3.2 Case with $R_y/R_x = -1$, Hyperbolic Paraboloidal Shell 283
10.3.3 Effect of Different Curvature 285
10.4 Buckling of Simply Supported Shells under Static Load 286
10.5 Theoretical Approach for Clamped Laminated Shells 286
10.6 Numerical Results for Vibrations of Clamped Laminated Shells 289
10.7 Buckling of the Space Shuttle Liquid-Oxygen Tank 291
References 296

11 Meshless Discretization of Plates and Shells of Complex Shape by Using the R-Functions 298
11.1 Introduction 298
11.1.1 Literature Review 298
11.2 The R-Functions Method 299
11.2.1 Boundary Value Problems with Homogeneous Dirichlet Boundary Conditions 299
11.2.1.1 Example: Shell with Complex Shape 301
11.2.2 Boundary Value Problems with Inhomogeneous Dirichlet Boundary Conditions 303
11.2.3 Boundary Value Problems with Neumann and Mixed Boundary Conditions 303
11.2.4 Admissible Functions for Shells and Plates with Different Boundary Conditions 304
11.3 Numerical Results for a Shallow-Shell with Complex Shape 306
11.4 Experimental Results and Comparison 308
References 309

12 Vibrations of Circular Plates and Rotating Disks 311
12.1 Introduction 311
12.1.1 Literature Review 311
12.2 Linear Vibrations of Circular and Annular Plates 313
12.3 Nonlinear Vibrations of Circular Plates 314
12.3.1 Numerical Results 317
12.4 Nonlinear Vibrations of Disks Spinning Near a Critical Speed 317
12.4.1 Numerical Results 320
References 323

13 Nonlinear Stability of Circular Cylindrical Shells under Static and Dynamic Axial Loads 325
13.1 Introduction 325
13.1.1 Literature Review 325
13.2 Theoretical Approach 328
13.3 Numerical Results 329
13.3.1 Static Bifurcations 329
13.3.2 Dynamic Loads 332
References 336

14 Nonlinear Stability and Vibration of Circular Shells Conveying Fluid 338
14.1 Introduction 338
14.1.1 Literature Review 338
## Contents

14.2 Fluid-Structure Interaction for Flowing Fluid ........................................ 339  
14.2.1 Fluid Model ................................................................................ 339  
14.2.2 Shell Expansion ........................................................................... 340  
14.2.3 Fluid-Structure Interaction ............................................................ 341  
14.2.4 Nonlinear Equations of Motion with Galerkin Method .................. 342  
14.2.5 Energy Associated with Flow and Lagrange Equations ............... 342  
14.2.6 Solution of the Associated Eigenvalue Problem ........................... 345  
14.3 Numerical Results for Stability ....................................................... 346  
14.4 Comparison of Numerical and Experimental Stability Results ........ 348  
14.5 Numerical Results for Nonlinear Forced Vibrations ...................... 350  
14.5.1 Periodic Response ....................................................................... 350  
14.5.2 Unsteady and Chaotic Motion ..................................................... 351  
References ............................................................................................... 353  

15 Nonlinear Supersonic Flutter of Circular Cylindrical Shells  
with Imperfections ................................................................................... 355  
15.1 Introduction ...................................................................................... 355  
15.1.1 Literature Review ....................................................................... 356  
15.2 Theoretical Approach ...................................................................... 358  
15.2.1 Linear and Third-Order Piston Theory ......................................... 358  
15.2.2 Structural Model ......................................................................... 359  
15.3 Numerical Results .......................................................................... 361  
15.3.1 Linear Results ............................................................................. 361  
15.3.2 Nonlinear Results without Geometric Imperfections ............... 363  
15.3.3 Nonlinear Results with Geometric Imperfections ..................... 367  
References ............................................................................................... 371  

Index ........................................................................................................ 373