

Cambridge University Press

978-0-521-88223-1 - Gravitation: Foundations and Frontiers

T. Padmanabhan

Frontmatter

[More information](#)

## GRAVITATION FOUNDATIONS AND FRONTIERS

Covering all aspects of gravitation in a contemporary style, this advanced textbook is ideal for graduate students and researchers in all areas of theoretical physics.

The ‘Foundations’ section develops the formalism in six chapters, and uses it in the next four chapters to discuss four key applications – spherical space-times, black holes, gravitational waves and cosmology. The six chapters in the ‘Frontiers’ section describe cosmological perturbation theory, quantum fields in curved spacetime, and the Hamiltonian structure of general relativity, among several other advanced topics, some of which are covered in-depth for the first time in a textbook.

The modular structure of the book allows different sections to be combined to suit a variety of courses. More than 225 exercises are included to test and develop the readers’ understanding. There are also over 30 projects to help readers make the transition from the book to their own original research.

T. PADMANABHAN is a Distinguished Professor and Dean of Core Academic Programmes at the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune. He is a renowned theoretical physicist and cosmologist with nearly 30 years of research and teaching experience both in India and abroad. Professor Padmanabhan has published over 200 research papers and nine books, including six graduate-level textbooks. These include the *Structure Formation in the Universe* and *Theoretical Astrophysics*, a comprehensive three-volume course. His research work has won prizes from the Gravity Research Foundation (USA) five times, including the First Prize in 2008. In 2007 he received the Padma Shri, the medal of honour from the President of India in recognition of his achievements.

Cambridge University Press  
978-0-521-88223-1 - Gravitation: Foundations and Frontiers  
T. Padmanabhan  
Frontmatter  
[More information](#)

---

# GRAVITATION

## Foundations and Frontiers

T. PADMANABHAN

*IUCAA, Pune,  
India*



CAMBRIDGE  
UNIVERSITY PRESS

Cambridge University Press  
 978-0-521-88223-1 - Gravitation: Foundations and Frontiers  
 T. Padmanabhan  
 Frontmatter  
[More information](#)

CAMBRIDGE UNIVERSITY PRESS  
 Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,  
 São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press  
 The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)  
 Information on this title: [www.cambridge.org/9780521882231](http://www.cambridge.org/9780521882231)

© T. Padmanabhan 2010

This publication is in copyright. Subject to statutory exception  
 and to the provisions of relevant collective licensing agreements,  
 no reproduction of any part may take place without  
 the written permission of Cambridge University Press.

First published 2010

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing-in-Publication Data*

Padmanabhan, T. (Thanu), 1957-  
 Gravitation : foundations and frontiers / T. Padmanabhan.  
 p. cm.

ISBN 978-0-521-88223-1 (Hardback)

1. Gravitation—Textbooks. 2. Relativity (Physics)—Textbooks. 3. Gravitation—Problems,  
 exercises, etc. I. Title.

QC178.P226 2010

531'.14—dc22

2009038009

ISBN 978-0-521-88223-1 Hardback

Cambridge University Press has no responsibility for the persistence or  
 accuracy of URLs for external or third-party Internet websites referred to  
 in this publication, and does not guarantee that any content on such  
 websites is, or will remain, accurate or appropriate.

Cambridge University Press  
978-0-521-88223-1 - Gravitation: Foundations and Frontiers  
T. Padmanabhan  
Frontmatter  
[More information](#)

---

*Dedicated to the fellow citizens of India*

Contents

<i>List of exercises</i>	<i>page</i> xiii
<i>List of projects</i>	xix
<i>Preface</i>	xxi
<i>How to use this book</i>	xxvii
<b>1 Special relativity</b>	1
1.1 Introduction	1
1.2 The principles of special relativity	1
1.3 Transformation of coordinates and velocities	6
1.3.1 Lorentz transformation	8
1.3.2 Transformation of velocities	10
1.3.3 Lorentz boost in an arbitrary direction	11
1.4 Four-vectors	13
1.4.1 Four-velocity and acceleration	17
1.5 Tensors	19
1.6 Tensors as geometrical objects	23
1.7 Volume and surface integrals in four dimensions	26
1.8 Particle dynamics	29
1.9 The distribution function and its moments	35
1.10 The Lorentz group and Pauli matrices	45
<b>2 Scalar and electromagnetic fields in special relativity</b>	54
2.1 Introduction	54
2.2 External fields of force	54
2.3 Classical scalar field	55
2.3.1 Dynamics of a particle interacting with a scalar field	55
2.3.2 Action and dynamics of the scalar field	57
2.3.3 Energy-momentum tensor for the scalar field	60
2.3.4 Free field and the wave solutions	62

2.3.5	Why does the scalar field lead to an attractive force?	64
2.4	Electromagnetic field	66
2.4.1	Charged particle in an electromagnetic field	67
2.4.2	Lorentz transformation of electric and magnetic fields	71
2.4.3	Current vector	73
2.5	Motion in the Coulomb field	75
2.6	Motion in a constant electric field	79
2.7	Action principle for the vector field	81
2.8	Maxwell's equations	83
2.9	Energy and momentum of the electromagnetic field	90
2.10	Radiation from an accelerated charge	95
2.11	Larmor formula and radiation reaction	100
3	<b>Gravity and spacetime geometry: the inescapable connection</b>	107
3.1	Introduction	107
3.2	Field theoretic approaches to gravity	107
3.3	Gravity as a scalar field	108
3.4	Second rank tensor theory of gravity	113
3.5	The principle of equivalence and the geometrical description of gravity	125
3.5.1	Uniformly accelerated observer	126
3.5.2	Gravity and the flow of time	128
4	<b>Metric tensor, geodesics and covariant derivative</b>	136
4.1	Introduction	136
4.2	Metric tensor and gravity	136
4.3	Tensor algebra in curved spacetime	141
4.4	Volume and surface integrals	146
4.5	Geodesic curves	149
4.5.1	Properties of geodesic curves	154
4.5.2	Affine parameter and null geodesics	156
4.6	Covariant derivative	162
4.6.1	Geometrical interpretation of the covariant derivative	163
4.6.2	Manipulation of covariant derivatives	167
4.7	Parallel transport	170
4.8	Lie transport and Killing vectors	173
4.9	Fermi–Walker transport	181
5	<b>Curvature of spacetime</b>	189
5.1	Introduction	189
5.2	Three perspectives on the spacetime curvature	189
5.2.1	Parallel transport around a closed curve	189
5.2.2	Non-commutativity of covariant derivatives	192

	<i>Contents</i>	ix
	5.2.3 Tidal acceleration produced by gravity	196
5.3	Properties of the curvature tensor	200
	5.3.1 Algebraic properties	200
	5.3.2 Bianchi identity	203
	5.3.3 Ricci tensor, Weyl tensor and conformal transformations	204
5.4	Physics in curved spacetime	208
	5.4.1 Particles and photons in curved spacetime	209
	5.4.2 Ideal fluid in curved spacetime	210
	5.4.3 Classical field theory in curved spacetime	217
	5.4.4 Geometrical optics in curved spacetime	221
5.5	Geodesic congruence and Raychaudhuri's equation	224
	5.5.1 Timelike congruence	225
	5.5.2 Null congruence	228
	5.5.3 Integration on null surfaces	230
5.6	Classification of spacetime curvature	231
	5.6.1 Curvature in two dimensions	232
	5.6.2 Curvature in three dimensions	233
	5.6.3 Curvature in four dimensions	234
<b>6</b>	<b>Einstein's field equations and gravitational dynamics</b>	<b>239</b>
	6.1 Introduction	239
	6.2 Action and gravitational field equations	239
	6.2.1 Properties of the gravitational action	242
	6.2.2 Variation of the gravitational action	244
	6.2.3 A digression on an alternative form of action functional	247
	6.2.4 Variation of the matter action	250
	6.2.5 Gravitational field equations	258
	6.3 General properties of gravitational field equations	261
	6.4 The weak field limit of gravity	268
	6.4.1 Metric of a stationary source in linearized theory	271
	6.4.2 Metric of a light beam in linearized theory	276
	6.5 Gravitational energy-momentum pseudo-tensor	279
<b>7</b>	<b>Spherically symmetric geometry</b>	<b>293</b>
	7.1 Introduction	293
	7.2 Metric of a spherically symmetric spacetime	293
	7.2.1 Static geometry and Birkoff's theorem	296
	7.2.2 Interior solution to the Schwarzschild metric	304
	7.2.3 Embedding diagrams to visualize geometry	311
	7.3 Vaidya metric of a radiating source	313
	7.4 Orbits in the Schwarzschild metric	314
	7.4.1 Precession of the perihelion	318

7.4.2	Deflection of an ultra-relativistic particle	323
7.4.3	Precession of a gyroscope	326
7.5	Effective potential for orbits in the Schwarzschild metric	329
7.6	Gravitational collapse of a dust sphere	334
<b>8</b>	<b>Black holes</b>	340
8.1	Introduction	340
8.2	Horizons in spherically symmetric metrics	340
8.3	Kruskal–Szekeres coordinates	343
8.3.1	Radial infall in different coordinates	350
8.3.2	General properties of maximal extension	356
8.4	Penrose–Carter diagrams	358
8.5	Rotating black holes and the Kerr metric	365
8.5.1	Event horizon and infinite redshift surface	368
8.5.2	Static limit	372
8.5.3	Penrose process and the area of the event horizon	374
8.5.4	Particle orbits in the Kerr metric	378
8.6	Super-radiance in Kerr geometry	381
8.7	Horizons as null surfaces	385
<b>9</b>	<b>Gravitational waves</b>	399
9.1	Introduction	399
9.2	Propagating modes of gravity	399
9.3	Gravitational waves in a flat spacetime background	402
9.3.1	Effect of the gravitational wave on a system of particles	409
9.4	Propagation of gravitational waves in the curved spacetime	413
9.5	Energy and momentum of the gravitational wave	416
9.6	Generation of gravitational waves	422
9.6.1	Quadrupole formula for the gravitational radiation	427
9.6.2	Back reaction due to the emission of gravitational waves	429
9.7	General relativistic effects in binary systems	434
9.7.1	Gravitational radiation from binary pulsars	434
9.7.2	Observational aspects of binary pulsars	438
9.7.3	Gravitational radiation from coalescing binaries	443
<b>10</b>	<b>Relativistic cosmology</b>	452
10.1	Introduction	452
10.2	The Friedmann spacetime	452
10.3	Kinematics of the Friedmann model	457
10.3.1	The redshifting of the momentum	458
10.3.2	Distribution functions for particles and photons	461
10.3.3	Measures of distance	462



## Contents

xi

10.4	Dynamics of the Friedmann model	466
10.5	The de Sitter spacetime	479
10.6	Brief thermal history of the universe	483
10.6.1	Decoupling of matter and radiation	484
10.7	Gravitational lensing	487
10.8	Killing vectors and the symmetries of the space	493
10.8.1	Maximally symmetric spaces	494
10.8.2	Homogeneous spaces	496
<b>11</b>	<b>Differential forms and exterior calculus</b>	<b>502</b>
11.1	Introduction	502
11.2	Vectors and 1-forms	502
11.3	Differential forms	510
11.4	Integration of forms	513
11.5	The Hodge duality	516
11.6	Spin connection and the curvature 2-forms	519
11.6.1	Einstein–Hilbert action and curvature 2-forms	523
11.6.2	Gauge theories in the language of forms	526
<b>12</b>	<b>Hamiltonian structure of general relativity</b>	<b>530</b>
12.1	Introduction	530
12.2	Einstein’s equations in (1+3)-form	530
12.3	Gauss–Codazzi equations	535
12.4	Gravitational action in (1+3)-form	540
12.4.1	The Hamiltonian for general relativity	542
12.4.2	The surface term and the extrinsic curvature	545
12.4.3	Variation of the action and canonical momenta	547
12.5	Junction conditions	552
12.5.1	Collapse of a dust sphere and thin-shell	554
<b>13</b>	<b>Evolution of cosmological perturbations</b>	<b>560</b>
13.1	Introduction	560
13.2	Structure formation and linear perturbation theory	560
13.3	Perturbation equations and gauge transformations	562
13.3.1	Evolution equations for the source	569
13.4	Perturbations in dark matter and radiation	572
13.4.1	Evolution of modes with $\lambda \gg d_H$	573
13.4.2	Evolution of modes with $\lambda \ll d_H$ in the radiation dominated phase	574
13.4.3	Evolution in the matter dominated phase	577
13.4.4	An alternative description of the matter–radiation system	578
13.5	Transfer function for the matter perturbations	582

13.6	Application: temperature anisotropies of CMBR	584
13.6.1	The Sachs–Wolfe effect	586
<b>14</b>	<b>Quantum field theory in curved spacetime</b>	<b>591</b>
14.1	Introduction	591
14.2	Review of some key results in quantum field theory	591
14.2.1	Bogolyubov transformations and the particle concept	596
14.2.2	Path integrals and Euclidean time	598
14.3	Exponential redshift and the thermal spectrum	602
14.4	Vacuum state in the presence of horizons	605
14.5	Vacuum functional from a path integral	609
14.6	Hawking radiation from black holes	618
14.7	Quantum field theory in a Friedmann universe	625
14.7.1	General formalism	625
14.7.2	Application: power law expansion	628
14.8	Generation of initial perturbations from inflation	631
14.8.1	Background evolution	632
14.8.2	Perturbations in the inflationary models	634
<b>15</b>	<b>Gravity in higher and lower dimensions</b>	<b>643</b>
15.1	Introduction	643
15.2	Gravity in lower dimensions	644
15.2.1	Gravity and black hole solutions in $(1 + 2)$ dimensions	644
15.2.2	Gravity in two dimensions	646
15.3	Gravity in higher dimensions	646
15.3.1	Black holes in higher dimensions	648
15.3.2	Brane world models	648
15.4	Actions with holography	653
15.5	Surface term and the entropy of the horizon	663
<b>16</b>	<b>Gravity as an emergent phenomenon</b>	<b>670</b>
16.1	Introduction	670
16.2	The notion of an emergent phenomenon	671
16.3	Some intriguing features of gravitational dynamics	673
16.3.1	Einstein’s equations as a thermodynamic identity	673
16.3.2	Gravitational entropy and the boundary term in the action	676
16.3.3	Horizon thermodynamics and Lanczos–Lovelock theories	677
16.4	An alternative perspective on gravitational dynamics	679
	<i>Notes</i>	689
	<i>Index</i>	695

# List of exercises

1.1	Light clocks	9
1.2	Superluminal motion	11
1.3	The strange world of four-vectors	16
1.4	Focused to the front	16
1.5	Transformation of antisymmetric tensors	23
1.6	Practice with completely antisymmetric tensors	23
1.7	A null curve in flat spacetime	29
1.8	Shadows are Lorentz invariant	29
1.9	Hamiltonian form of action – Newtonian mechanics	34
1.10	Hamiltonian form of action – special relativity	34
1.11	Hitting a mirror	34
1.12	Photon–electron scattering	35
1.13	More practice with collisions	35
1.14	Relativistic rocket	35
1.15	Practice with equilibrium distribution functions	44
1.16	Projection effects	44
1.17	Relativistic virial theorem	44
1.18	Explicit computation of spin precession	52
1.19	Little group of the Lorentz group	52
2.1	Measuring the $F^a_b$	70
2.2	Schrödinger equation and gauge transformation	70
2.3	Four-vectors leading to electric and magnetic fields	70
2.4	Hamiltonian form of action – charged particle	71
2.5	Three-dimensional form of the Lorentz force	71
2.6	Pure gauge imposters	71
2.7	Pure electric or magnetic fields	74
2.8	Elegant solution to non-relativistic Coulomb motion	77
2.9	More on uniformly accelerated motion	80

xiv	<i>List of exercises</i>	
2.10	Motion of a charge in an electromagnetic plane wave	80
2.11	Something to think about: swindle in Fourier space?	83
2.12	Hamiltonian form of action – electromagnetism	88
2.13	Eikonal approximation	88
2.14	General solution to Maxwell’s equations	88
2.15	Gauge covariant derivative	89
2.16	Massive vector field	90
2.17	What is $c$ if there are no massless particles?	90
2.18	Conserving the total energy	93
2.19	Stresses and strains	93
2.20	Everything obeys Einstein	93
2.21	Practice with the energy-momentum tensor	93
2.22	Poynting–Robertson effect	94
2.23	Moving thermometer	95
2.24	Standard results about radiation	100
2.25	Radiation drag	103
3.1	Motion of a particle in scalar theory of gravity	112
3.2	Field equations of the tensor theory of gravity	122
3.3	Motion of a particle in tensor theory of gravity	123
3.4	Velocity dependence of effective charge for different spins	124
3.5	Another form of the Rindler metric	128
3.6	Alternative derivation of the Rindler metric	128
4.1	Practice with metrics	141
4.2	Two ways of splitting spacetimes into space and time	152
4.3	Hamiltonian form of action – particle in curved spacetime	153
4.4	Gravo-magnetic force	153
4.5	Flat spacetime geodesics in curvilinear coordinates	155
4.6	Gaussian normal coordinates	156
4.7	Non-affine parameter: an example	160
4.8	Refractive index of gravity	160
4.9	Practice with the Christoffel symbols	161
4.10	Vanishing Hamiltonians	161
4.11	Transformations that leave geodesics invariant	161
4.12	Accelerating without moving	163
4.13	Covariant derivative of tensor densities	169
4.14	Parallel transport on a sphere	172
4.15	Jacobi identity	179
4.16	Understanding the Lie derivative	180
4.17	Understanding the Killing vectors	180
4.18	Killing vectors for a gravitational wave metric	181

*List of exercises* xv

4.19	Tetrad for a uniformly accelerated observer	183
5.1	Curvature in the Newtonian approximation	195
5.2	Non-geodesic deviation	199
5.3	Measuring the curvature tensor	199
5.4	Spinning body in curved spacetime	199
5.5	Explicit transformation to the Riemann normal coordinates	202
5.6	Curvature tensor in the language of gauge fields	203
5.7	Conformal transformations and curvature	206
5.8	Splitting the spacetime and its curvature	207
5.9	Matrix representation of the curvature tensor	207
5.10	Curvature in synchronous coordinates	208
5.11	Pressure gradient needed to support gravity	215
5.12	Thermal equilibrium in a static metric	216
5.13	Weighing the energy	216
5.14	General relativistic Bernoulli equation	216
5.15	Conformal invariance of electromagnetic action	219
5.16	Gravity as an optically active media	219
5.17	Curvature and Killing vectors	220
5.18	Christoffel symbols and infinitesimal diffeomorphism	220
5.19	Conservation of canonical momentum	220
5.20	Energy-momentum tensor and geometrical optics	224
5.21	Ray optics in Newtonian approximation	224
5.22	Expansion and rotation of congruences	231
5.23	Euler characteristic of two-dimensional spaces	233
6.1	Palatini variational principle	246
6.2	Connecting Einstein gravity with the spin-2 field	247
6.3	Action with Gibbons–Hawking–York counterterm	250
6.4	Electromagnetic current from varying the action	252
6.5	Geometrical interpretation of the spin-2 field	256
6.6	Conditions on the energy-momentum tensor	257
6.7	Pressure as the Lagrangian for a fluid	257
6.8	Generic decomposition of an energy-momentum tensor	257
6.9	Something to think about: disaster if we vary $g_{ab}$ rather than $g^{ab}$ ?	260
6.10	Newtonian approximation with cosmological constant	260
6.11	Wave equation for $F_{mn}$ in curved spacetime	265
6.12	Structure of the gravitational action principle	268
6.13	Deflection of light in the Newtonian approximation	277
6.14	Metric perturbation due to a fast moving particle	277
6.15	Metric perturbation due to a non-relativistic source	278
6.16	Landau–Lifshitz pseudo-tensor in the Newtonian approximation	284

xvi	<i>List of exercises</i>	
6.17	More on the Landau–Lifshitz pseudo-tensor	284
6.18	Integral for the angular momentum	285
6.19	Several different energy-momentum pseudo-tensors	285
6.20	Alternative expressions for the mass	288
7.1	A reduced action principle for spherical geometry	295
7.2	Superposition in spherically symmetric spacetimes	301
7.3	Reissner–Nordstrom metric	302
7.4	Spherically symmetric solutions with a cosmological constant	302
7.5	Time dependent spherically symmetric metric	303
7.6	Schwarzschild metric in a different coordinate system	304
7.7	Variational principle for pressure support	309
7.8	Internal metric of a constant density star	310
7.9	Clock rates on the surface of the Earth	310
7.10	Metric of a cosmic string	310
7.11	Static solutions with perfect fluids	311
7.12	Model for a neutron star	312
7.13	Exact solution of the orbit equation in terms of elliptic functions	322
7.14	Contribution of nonlinearity to perihelion precession	322
7.15	Perihelion precession for an oblate Sun	322
7.16	Angular shift of the direction of stars	324
7.17	Time delay for photons	324
7.18	Deflection of light in the Schwarzschild–de Sitter metric	325
7.19	Solar corona and the deflection of light by the Sun	325
7.20	General expression for relativistic precession	328
7.21	Hafele–Keating experiment	328
7.22	Exact solution of the orbital equation	331
7.23	Effective potential for the Reissner–Nordstrom metric	332
7.24	Horizons are forever	332
7.25	Redshift of the photons	333
7.26	Going into a shell	333
7.27	You look fatter than you are	334
7.28	Capture of photons by a Schwarzschild black hole	334
7.29	Twin paradox in the Schwarzschild metric?	334
7.30	Spherically symmetric collapse of a scalar field	337
8.1	The weird dynamics of an eternal black hole	349
8.2	Dropping a charge into the Schwarzschild black hole	355
8.3	Painlevé coordinates for the Schwarzschild metric	355
8.4	Redshifts of all kinds	356
8.5	Extreme Reissner–Nordstrom solution	363
8.6	Multisource extreme black hole solution	364
8.7	A special class of metric	368

*List of exercises* xvii

8.8	Closed timelike curves in the Kerr metric	371
8.9	Zero angular momentum observers (ZAMOs)	374
8.10	Circular orbits in the Kerr metric	380
8.11	Killing tensor	381
8.12	Practice with null surfaces and local Rindler frames	389
8.13	Zeroth law of black hole mechanics	394
9.1	Gravity wave in the Fourier space	408
9.2	Effect of rotation on a $TT$ gravitational wave	408
9.3	Not every perturbation can be $TT$	409
9.4	Nevertheless it moves – in a gravitational wave	413
9.5	The optics of gravitational waves	416
9.6	The $R_{ambn}^{(1)}$ is not gauge invariant, but ...	416
9.7	An exact gravitational wave metric	416
9.8	Energy-momentum tensor of the gravitational wave from the spin-2 field	421
9.9	Landau–Lifshitz pseudo-tensor for the gravitational wave	421
9.10	Gauge dependence of the energy of the gravitational waves	422
9.11	The $TT$ part of the gravitational radiation from first principles	426
9.12	Flux of gravitational waves	428
9.13	Original issues	429
9.14	Absorption of gravitational waves	429
9.15	Lessons from gravity for electromagnetism	433
9.16	Eccentricity matters	437
9.17	Getting rid of eccentric behaviour	437
9.18	Radiation from a parabolic trajectory	438
9.19	Gravitational waves from a circular orbit	438
9.20	Pulsar timing and the gravitational wave background	442
10.1	Friedmann model in spherically symmetric coordinates	456
10.2	Conformally flat form of the metric	457
10.3	Particle velocity in the Friedmann universe	460
10.4	Geodesic equation in the Friedmann universe	460
10.5	Generalized formula for photon redshift	460
10.6	Electromagnetism in the closed Friedmann universe	461
10.7	Nice features of the conformal time	475
10.8	Tracker solutions for scalar fields	476
10.9	Horizon size	476
10.10	Loitering and other universes	477
10.11	Point particle in a Friedmann universe	477
10.12	Collapsing dust ball revisited	477
10.13	The anti-de Sitter spacetime	482

xviii	<i>List of exercises</i>	
10.14	Geodesics in de Sitter spacetime	482
10.15	Poincaré half-plane	495
10.16	The Gödel universe	495
10.17	Kasner model of the universe	499
10.18	CMBR in a Bianchi Type I model	500
11.1	Frobenius theorem in the language of forms	512
11.2	The Dirac string	516
11.3	Simple example of a non-exact, closed form	516
11.4	Dirac equation in curved spacetime	523
11.5	Bianchi identity in the form language	525
11.6	Variation of Einstein–Hilbert action in the form language	525
11.7	The Gauss–Bonnet term	525
11.8	Landau–Lifshitz pseudo-tensor in the form language	526
11.9	Bianchi identity for gauge fields	528
11.10	Action and topological invariants in the gauge theory	528
12.1	Extrinsic curvature and covariant derivative	535
12.2	Gauss–Codazzi equations for a cone and a sphere	540
12.3	Matching conditions	557
12.4	Vacuole in a dust universe	557
13.1	Synchronous gauge	567
13.2	Gravitational waves in a Friedmann universe	568
13.3	Perturbed Einstein tensor in an arbitrary gauge	568
13.4	Mészáros solution	577
13.5	Growth factor in an open universe	582
13.6	Cosmic variance	585
14.1	Path integral kernel for the harmonic oscillator	602
14.2	Power spectrum of a wave with exponential redshift	604
14.3	Casimir effect	608
14.4	Bogolyubov coefficients for (1+1) Rindler coordinates	615
14.5	Bogolyubov coefficients for (1+3) Rindler coordinates	615
14.6	Rindler vacuum and the analyticity of modes	616
14.7	Response of an accelerated detector	617
14.8	Horizon entropy and the surface term in the action	624
14.9	Gauge invariance of $\mathcal{R}$	640
14.10	Coupled equations for the scalar field perturbations	640
15.1	Field equations in the Gauss–Bonnet theory	659
15.2	Black hole solutions in the Gauss–Bonnet theory	659
15.3	Analogue of Bianchi identity in the Lanczos–Lovelock theories	660
15.4	Entropy as the Noether charge	667
16.1	Gauss–Bonnet field equations as a thermodynamic identity	678



## List of projects

1.1	Energy-momentum tensor of non-ideal fluids	52
2.1	Third rank tensor field	104
2.2	Hamilton–Jacobi structure of electrodynamics	104
2.3	Does a uniformly accelerated charge radiate?	105
3.1	Self-coupled scalar field theory of gravity	131
3.2	Is there hope for scalar theories of gravity?	132
3.3	Attraction of light	132
3.4	Metric corresponding to an observer with variable acceleration	133
3.5	Schwinger’s magic	133
4.1	Velocity space metric	186
4.2	Discovering gauge theories	187
5.1	Parallel transport, holonomy and curvature	236
5.2	Point charge in the Schwarzschild metric	237
6.1	Scalar tensor theories of gravity	288
6.2	Einstein’s equations for a stationary metric	290
6.3	Holography of the gravitational action	291
7.1	Embedding the Schwarzschild metric in six dimensions	338
7.2	Poor man’s approach to the Schwarzschild metric	338
7.3	Radiation reaction in curved spacetime	339
8.1	Noether’s theorem and the black hole entropy	394
8.2	Wave equation in a black hole spacetime	396
8.3	Quasi-normal modes	397
9.1	Gauge and dynamical degrees of freedom	446
9.2	An exact gravitational plane wave	447
9.3	Post-Newtonian approximation	448
10.1	Examples of gravitational lensing	500
12.1	Superspace and the Wheeler–DeWitt equation	557
13.1	Nonlinear perturbations and cosmological averaging	588

xx	<i>List of projects</i>	
14.1	Detector response in stationary trajectories	641
14.2	Membrane paradigm for the black holes	641
14.3	Accelerated detectors in curved spacetime	642
15.1	Boundary terms for the Lanczos–Lovelock action	668

## Preface

There is a need for a *comprehensive, advanced level*, textbook dealing with all aspects of gravity, written *for the physicist* in a *contemporary* style. The italicized adjectives in the above sentence are the key: most of the existing books on the market are either outdated in emphasis, too mathematical for a physicist, not comprehensive or written at an elementary level. (For example, the two unique books – L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, and C. W. Misner, K. S. Thorne and J. A. Wheeler (MTW), *Gravitation* – which I consider to be masterpieces in this subject are more than three decades old and are out of date in their emphasis.) The current book is expected to fill this niche and I hope it becomes a standard reference in this field. Some of the features of this book, including the summary of chapters, are given below.

As the title implies, this book covers both Foundations (Chapters 1–10) and Frontiers (Chapters 11–16) of general relativity so as to cater for the needs of different segments of readership. The Foundations acquaint the readers with the basics of general relativity while the topics in Frontiers allow one to ‘mix-and-match’, depending on interest and inclination. This modular structure of the book will allow it to be adapted for different types of course work.

For a specialist researcher or a student of gravity, this book provides a comprehensive coverage of all the contemporary topics, some of which are discussed in a textbook for the first time, as far as I know. The cognoscenti will find that there is a fair amount of originality in the discussion (and in the Exercises) of even the conventional topics.

While the book is quite comprehensive, it also has a structure which will make it accessible to a wide target audience. Researchers and teachers interested in theoretical physics, general relativity, relativistic astrophysics and cosmology will find it useful for their research and adaptable for their course requirements. (The section *How to use this book*, just after this Preface, gives more details of this aspect.) The discussion is presented in a style suitable for physicists, ensuring that it caters

for the current interest in gravity among physicists working in related areas. The large number (more than 225) of reasonably nontrivial Exercises makes it ideal for self-study.

Another unique feature of this book is a set of Projects at the end of selected chapters. The Projects are advanced level exercises presented with helpful hints to show the reader a direction of attack. Several of them are based on research literature dealing with key open issues in different areas. These will act as a bridge for students to cross over from textbook material to real life research. Graduate students and grad school teachers will find the Exercises and Projects extremely useful. Advanced undergraduate students with a flair for theoretical physics will also be able to use parts of this book, especially in combination with more elementary books.

Here is a brief description of the chapters of the book and their inter-relationship.

Chapters 1 and 2 of this book are somewhat unique and serve an important purpose, which I would like to explain. A student learning general relativity often finds that she simultaneously has to cope with (i) conceptual and mathematical issues which arise from the spacetime being curved and (ii) technical issues and concepts which are essentially special relativistic but were never emphasized adequately in a special relativity course! For example, manipulation of four-dimensional integrals or the concept and properties of the energy-momentum tensor have nothing to do with general relativity *a priori* – but are usually not covered in depth in conventional special relativity courses. The first two chapters give the student a rigorous training in four-dimensional techniques in flat spacetime so that she can concentrate on issues which are genuinely general relativistic later on. These chapters can also usefully serve as modular course material for a short course on advanced special relativity or classical field theory.

Chapter 1 introduces special relativity using four-vectors and the action principle right from the outset. Chapter 2 introduces the electromagnetic field through the four-vector formalism. I expect the student to have done a standard course in classical mechanics and electromagnetic theory but I do *not* assume familiarity with the relativistic (four-vector) notation. Several topics that are needed later in general relativity are introduced in these two chapters in order to familiarize the reader early on. Examples include the use of the relativistic Hamilton–Jacobi equation, precession of Coulomb orbits, dynamics of the electromagnetic field obtained from an action principle, derivation of the field of an arbitrarily moving charged particle, radiation reaction, etc. Chapter 2 also serves as a launch pad for discussing spin-0 and spin-2 interactions, using electromagnetism as a familiar example.

Chapter 3 attempts to put together special relativity and gravity and explains, in clear and precise terms, why it does not lead to a consistent picture. Most textbooks I know (except MTW) do not explain the issues involved clearly and with adequate

detail. For example, this chapter contains a detailed discussion of the spin-2 tensor field which is not available in textbooks. It is important for a student to realize that the description of gravity in terms of curvature of spacetime is inevitable and natural. This chapter will also lay the foundation for the description of the spin-2 tensor field  $h_{ab}$ , which will play an important role in the study of gravitational waves and cosmological perturbation theory later on.

Having convinced the reader that gravity is related to spacetime geometry, Chapter 4 begins with the description of general relativity by introducing the metric tensor and extending the ideas of four-vectors, tensors, etc., to a nontrivial background. There are two points that I would like to highlight about this chapter. First, I have introduced every concept with a physical principle rather than in the abstract language of differential geometry. For example, direct variation of the line interval leads to the geodesic equation *through* which one can motivate the notion of Christoffel symbols, covariant derivative, etc., in a simple manner. During the courses I have taught over years, students have found this approach attractive and simpler to grasp. Second, I find that students sometimes confuse issues which arise when curvilinear coordinates are used in flat spacetime with those related to genuine curvature. This chapter clarifies such issues.

Chapter 5 introduces the concept of the curvature tensor from three different perspectives and describes its properties. It then moves on to provide a complete description of electrodynamics, statistical mechanics, thermodynamics and wave propagation in curved spacetime, including the Raychaudhuri equation and the focusing theorem.

Chapter 6 starts with a clear and coherent derivation of Einstein's field equations from an action principle. I have provided a careful discussion of the surface term in the Einstein–Hilbert action (again not usually found in textbooks) in a manner which is quite general and turns out to be useful in the discussion of Lanczos–Lovelock models in Chapter 15. I then proceed to discuss the general structure of the field equations, the energy-momentum pseudo-tensor for gravity and the weak field limit of gravity.

After developing the formalism in the first six chapters, I apply it to discuss four key applications of general relativity – spherically symmetric spacetimes, black hole physics, gravitational waves and cosmology – in the next four chapters. (The only other key topic I have omitted, due to lack of space, is the physics of compact stellar remnants.)

Chapter 7 deals with the simplest class of exact solutions to Einstein's equations, which are those with spherical symmetry. The chapter also discusses the orbits of particles and photons in these spacetimes and the tests of general relativity. These are used in Chapter 8, which covers several aspects of black hole physics, concentrating mostly on the Schwarzschild and Kerr black holes. It also introduces

important concepts like the maximal extension of a manifold, Penrose–Carter diagrams and the geometrical description of horizons as null surfaces. A derivation of the zeroth law of black hole mechanics and illustrations of the first and second laws are also provided. The material developed here forms the backdrop for the discussions in Chapters 13, 15 and 16.

Chapter 9 takes up one of the key new phenomena that arise in general relativity, viz. the existence of solutions to Einstein’s equations which represent disturbances in the spacetime that propagate at the speed of light. A careful discussion of gauge invariance and coordinate conditions in the description of gravitational waves is provided. I also make explicit contact with similar phenomena in the case of electromagnetic radiation in order to help the reader to understand the concepts better. A detailed discussion of the binary pulsar is included and a Project at the end of the chapter explores the nuances of the post-Newtonian approximation.

Chapter 10 applies general relativity to study cosmology and the evolution of the universe. Given the prominence cosmology enjoys in current research and the fact that this interest will persist in future, it is important that all general relativists are acquainted with cosmology at the same level of detail as, for example, with the Schwarzschild metric. This is the motivation for Chapter 10 as well as Chapter 13 (which deals with general relativistic perturbation theory). The emphasis here will be mostly on the geometrical aspects of the universe rather than on physical cosmology, for which several other excellent textbooks (e.g. mine!) exist. However, in order to provide a complete picture and to appreciate the interplay between theory and observation, it is necessary to discuss certain aspects of the evolutionary history of the universe – which is done to the extent needed.

The second part of the book (Frontiers, Chapters 11–16) discusses six separate topics which are reasonably independent of each other (though not completely). While a student or researcher specializing in gravitation should study all of them, others could choose the topics based on their interest after covering the first part of the book.

Chapter 11 introduces the language of differential forms and exterior calculus and translates many of the results of the previous chapters into the language of forms. It also describes briefly the structure of gauge theories to illustrate the generality of the formalism. The emphasis is in developing the key concepts rapidly and connecting them up with the more familiar language used in the earlier chapters, rather than in maintaining mathematical rigour.

Chapter 12 describes the  $(1 + 3)$ -decomposition of general relativity and its Hamiltonian structure. I provide a derivation of Gauss–Codazzi equations and Einstein’s equations in the  $(1 + 3)$ -form. The connection between the surface term in the Einstein–Hilbert action and the extrinsic curvature of the boundary is also

spelt out in detail. Other topics include the derivation of junction conditions which are used later in Chapter 15 while discussing the brane world cosmologies.

Chapter 13 describes general relativistic linear perturbation theory in the context of cosmology. This subject has acquired major significance, thanks to the observational connection it makes with cosmic microwave background radiation. In view of this, I have also included a brief discussion of the application of perturbation theory in deriving the temperature anisotropies of the background radiation.

Chapter 14 describes some interesting results which arise when one studies standard quantum field theory in a background spacetime with a nontrivial metric. Though the discussion is reasonably self-contained, some familiarity with simple ideas of quantum theory of free fields will be helpful. The key result which I focus on is the intriguing connection between thermodynamics and horizons. This connection can be viewed from very different perspectives not all of which can rigorously be proved to be equivalent to one another. In view of the importance of this result, most of this chapter concentrates on obtaining this result using different techniques and interpreting it physically. In the latter part of the chapter, I have added a discussion of quantum field theory in the Friedmann universe and the generation of perturbations during the inflationary phase of the universe.

Chapter 15 discusses a few *selected* topics in the study of gravity in dimensions other than  $D = 4$ . I have kept the discussion of models in  $D < 4$  quite brief and have spent more time on the  $D > 4$  case. In this context – after providing a brief, but adequate, discussion of brane world models which are enjoying some popularity currently – I describe the structure of Lanczos–Lovelock models in detail. These models share several intriguing features with Einstein’s theory and constitute a natural generalization of Einstein’s theory to higher dimensions. I hope this chapter will fill the need, often felt by students working in this area, for a textbook discussion of Lanczos–Lovelock models.

The final chapter provides a perspective on gravity as an emergent phenomenon. (Obviously, this chapter shows my personal bias but I am sure that is acceptable in the *last* chapter!) I have tried to put together several peculiar features in the standard description of gravity and emphasize certain ideas which the reader might find fascinating and intriguing.

Because of the highly pedagogical nature of the material covered in this textbook, I have not given detailed references to original literature except on rare occasions when a particular derivation is not available in the standard textbooks. The annotated list of Notes given at the end of the book cites several other text books which I found useful. Some of these books contain extensive bibliographies and references to original literature. The selection of books and references cited here clearly reflects the bias of the author and I apologize to anyone who feels their work or contribution has been overlooked.

Discussions with several people, far too numerous to name individually, have helped me in writing this book. Here I shall confine myself to those who provided detailed comments on earlier drafts of the manuscript. Donald Lynden-Bell and Aseem Paranjape provided extensive and very detailed comments on most of the chapters and I am very thankful to them. I also thank A. Das, S. Dhurandar, P. P. Divakaran, J. Ehlers, G. F. R. Ellis, Rajesh Gopakumar, N. Kumar, N. Mukunda, J. V. Narlikar, Maulik Parikh, T. R. Seshadri and L. Sriramkumar for detailed comments on selected chapters.

Vince Higgs (CUP) took up my proposal to write this book with enthusiasm. The processing of this volume was handled by Laura Clark (CUP) and I thank her for the effort she has put in.

This project would not have been possible without the dedicated support from Vasanthi Padmanabhan, who not only did the entire TEXing and formatting but also produced most of the figures. I thank her for her help. It is a pleasure to acknowledge the library and other research facilities available at IUCAA, which were useful in this task.



## How to use this book

This book can be adapted by readers with varying backgrounds and requirements as well as by teachers handling different courses. The material is presented in a fairly modular fashion and I describe below different sub-units that can be combined for possible courses or for self-study.

### **1 Advanced special relativity**

Chapter 1 along with parts of Chapter 2 (especially Sections 2.2, 2.5, 2.6, 2.10) can form a course in advanced special relativity. No previous familiarity with four-vector notation (in the description of relativistic mechanics or electrodynamics) is required.

### **2 Classical field theory**

Parts of Chapter 1 along with Chapter 2 and Sections 3.2, 3.3 will give a comprehensive exposure to classical field theory. This will require familiarity with special relativity using four-vector notation which can be acquired from specific sections of Chapter 1.

### **3 Introductory general relativity**

Assuming familiarity with special relativity, a basic course in general relativity (GR) can be structured using the following material: Sections 3.5, Chapter 4 (except Sections 4.8, 4.9), Chapter 5 (except Sections 5.2.3, 5.3.3, 5.4.4, 5.5, 5.6), Sections 6.2.5, 6.4.1, 7.2.1, 7.4.1, 7.4.2, 7.5. This can be supplemented with selected topics in Chapters 8 and 9.

### **4 Relativistic cosmology**

Chapter 10 (except Sections 10.6, 10.7) along with Chapter 13 and parts of Sections 14.7 and 14.8 will constitute a course in relativistic cosmology and perturbation theory from a contemporary point of view.

### **5 Quantum field theory in curved spacetime**

Parts of Chapter 8 (especially Sections 8.2, 8.3, 8.7) and Chapter 14 will constitute a first course in this subject. It will assume familiarity with GR but not with detailed properties of black holes or quantum field theory. Parts of Chapter 2 can supplement this course.

## 6 Applied general relativity

For students who have already done a first course in GR, Chapters 6, 8, 9 and 12 (with parts of Chapter 7 not covered in the first course) will provide a description of *advanced* topics in GR.

### *Exercises and Projects*

None of the Exercises in this book is trivial or of simple ‘plug-in’ type. Some of them involve extending the concepts developed in the text or understanding them from a different perspective; others require detailed application of the material introduced in the chapter. There are more than 225 exercises and it is strongly recommended that the reader attempts as many as possible. Some of the nontrivial exercises contain hints and short answers.

The Projects are more advanced exercises linking to original literature. It will often be necessary to study additional references in order to comprehensively grasp or answer the questions raised in the projects. Many of them are open-ended (and could even lead to publishable results) but all of them are presented in a graded manner so that a serious student will be able to complete most parts of any project. They are included so as to provide a bridge for students to cross over from the textbook material to original research and should be approached in this light.

### *Notation and conventions*

Throughout the book, the Latin indices  $a, b, \dots, i, j, \dots$ , etc., run over 0, 1, 2, 3 with the 0-index denoting the time dimension and (1, 2, 3) denoting the standard space dimensions. The Greek indices,  $\alpha, \beta, \dots$ , etc., will run over 1, 2, 3. Except when indicated otherwise, the units are chosen with  $c = 1$ .

We will use the vector notation for both three-vectors and four-vectors by using different fonts. The four-momentum, for example, will be denoted by  $\mathbf{p}$  while the three-momentum will be denoted by  $\mathbf{p}$ .

The signature is  $(-, +, +, +)$  and curvature tensor is defined by the convention  $R^a_{bcd} \equiv \partial_c \Gamma^a_{bd} - \dots$  with  $R_{bd} = R^a_{bad}$ .

The symbol  $\equiv$  is used to indicate that the equation defines a new variable or notation.