1 Stress and Strain

An understanding of stress and strain is essential for analyzing metal forming operations. Often the words *stress* and *strain* are used synonymously by the nonscientific public. In engineering, however, stress is the intensity of force and strain is a measure of the amount of deformation.

1.1 STRESS

Stress is defined as the intensity of force, F, at a point.

$$\sigma = \partial F / \partial A \quad \text{as} \quad \partial A \to 0, \tag{1.1}$$

where A is the area on which the force acts.

If the stress is the same everywhere,

$$\sigma = F/A. \tag{1.2}$$

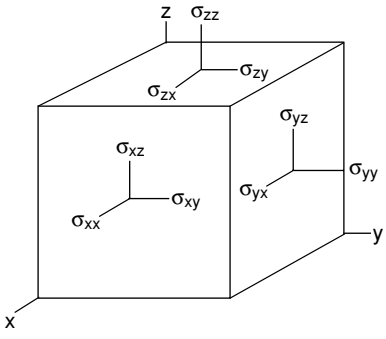
There are nine components of stress as shown in Figure 1.1. A normal stress component is one in which the force is acting normal to the plane. It may be tensile or compressive. A shear stress component is one in which the force acts parallel to the plane.

Stress components are defined with two subscripts. The first denotes the normal to the plane on which the force acts and the second is the direction of the force.* For example, σ_{xx} is a tensile stress in the x-direction. A shear stress acting on the x-plane in the y-direction is denoted σ_{xy}.

Repeated subscripts (e.g., $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$) indicate normal stresses. They are tensile if both subscripts are positive or both are negative. If one is positive and the other is negative, they are compressive. Mixed subscripts (e.g., $\sigma_{zx}, \sigma_{xy}, \sigma_{yz}$) denote shear stresses. A state of stress in tensor notation is expressed as

$$\sigma_{ij} = \begin{vmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix}, \tag{1.3}$$

* The use of the opposite convention should cause no problem because $\sigma_{ij} = \sigma_{ji}$.



1.1. Nine components of stress acting on an infinitesimal element.

where i and j are iterated over x, y, and z. Except where tensor notation is required, it is simpler to use a single subscript for a normal stress and denote a shear stress by τ. For example, $\sigma_x \equiv \sigma_{xx}$ and $\tau_{xy} \equiv \sigma_{xy}$.

1.2 STRESS TRANSFORMATION

Stress components expressed along one set of axes may be expressed along any other set of axes. Consider resolving the stress component $\sigma_y = F_y/A_y$ onto the x' and y' axes as shown in Figure 1.2.

The force F'_y in the y' direction is $F'_y = F_y \cos\theta$ and the area normal to y' is $A_{y'} = A_y/\cos\theta$, so

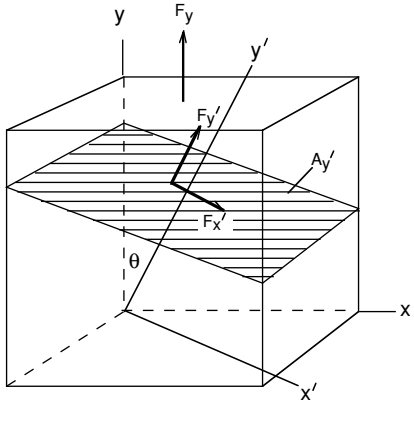
$$\sigma_{y'} = F_{y'}/A_{y'} = F_y \cos\theta/(A_y/\cos\theta) = \sigma_y \cos^2\theta. \tag{1.4a}$$

Similarly

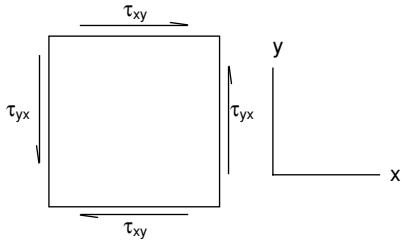
$$\tau_{y'x'} = F_{x'}/A_{y'} = F_y \sin\theta/(A_y/\cos\theta) = \sigma_y \cos\theta \sin\theta. \tag{1.4b}$$

Note that transformation of stresses requires two sine and/or cosine terms.

Pairs of shear stresses with the same subscripts in reverse order are always equal (e.g., $\tau_{ij} = \tau_{ji}$). This is illustrated in Figure 1.3 by a simple moment balance on an



1.2. The stresses acting on a plane, A', under a normal stress, σ_y.



1.3. Unless $\tau_{xy} = \tau_{yx}$, there would not be a moment balance.

infinitesimal element. Unless $\tau_{ij} = \tau_{ji}$, there would be an infinite rotational acceleration. Therefore

$$\tau_{ij} = \tau_{ji}. \tag{1.5}$$

The general equation for transforming the stresses from one set of axes (e.g., n, m, p) to another set of axes (e.g., i, j, k) is

$$\sigma_{ij} = \sum_{n=1}^{3}\sum_{m=1}^{3} \ell_{im}\ell_{jn}\sigma_{mn}. \tag{1.6}$$

Here, the term ℓ_{im} is the cosine of the angle between the i and m axes and the term ℓ_{jn} is the cosine of the angle between the j and n axes. This is often written as

$$\sigma_{ij} = \ell_{im}\ell_{jn}\sigma_{mn}, \tag{1.7}$$

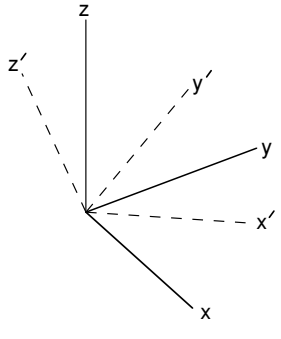
with the summation implied. Consider transforming stresses from the x, y, z axis system to the x', y', z' system shown in Figure 1.4.

Using equation 1.6,

$$\begin{aligned}\sigma_{x'x'} = {} & \ell_{x'x}\ell_{x'x}\sigma_{xx} + \ell_{x'x}\ell_{x'y}\sigma_{xy} + \ell_{x'x}\ell_{x'z}\sigma_{xz} \\ & + \ell_{x'y}\ell_{x'x}\sigma_{yx} + \ell_{x'y}\ell_{x'y}\sigma_{yy} + \ell_{x'y}\ell_{x'z}\sigma_{yz} \\ & + \ell_{x'z}\ell_{x'x}\sigma_{zx} + \ell_{x'z}\ell_{x'y}\sigma_{yz} + \ell_{x'z}\ell_{x'z}\sigma_{zz}\end{aligned} \tag{1.8a}$$

and

$$\begin{aligned}\sigma_{x'y'} = {} & \ell_{x'x}\ell_{y'x}\sigma_{xx} + \ell_{x'x}\ell_{y'y}\sigma_{xy} + \ell_{x'x}\ell_{y'z}\sigma_{xz} \\ & + \ell_{x'y}\ell_{y'x}\sigma_{yx} + \ell_{x'y}\ell_{y'y}\sigma_{yy} + \ell_{x'y}\ell_{y'z}\sigma_{yz} \\ & + \ell_{x'z}\ell_{y'x}\sigma_{zx} + \ell_{x'z}\ell_{y'y}\sigma_{yz} + \ell_{x'z}\ell_{y'z}\sigma_{zz}\end{aligned} \tag{1.8b}$$



1.4. Two orthogonal coordinate systems.

These can be simplified to

$$\sigma_{x'} = \ell_{x'x}^2\sigma_x + \ell_{x'y}^2\sigma_y + \ell_{x'z}^2\sigma_z + 2\ell_{x'y}\ell_{x'z}\tau_{yz} + 2\ell_{x'z}\ell_{x'x}\tau_{zx} + 2\ell_{x'x}\ell_{x'y}\tau_{xy} \tag{1.9a}$$

and

$$\begin{aligned}\tau_{x'y'} = {} & \ell_{x'x}\ell_{y'x}\sigma_x + \ell_{x'y}\ell_{y'y}\sigma_y + \ell_{x'z}\ell_{y'z}\sigma_z + (\ell_{x'y}\ell_{y'z} + \ell_{x'z}\ell_{y'y})\tau_{yz} \\ & + (\ell_{x'z}\ell_{y'x} + \ell_{x'x}\ell_{y'z})\tau_{zx} + (\ell_{x'x}\ell_{y'y} + \ell_{x'y}\ell_{y'x})\tau_{xy}.\end{aligned} \tag{1.9b}$$

1.3 PRINCIPAL STRESSES

It is always possible to find a set of axes along which the shear stress terms vanish. In this case σ_1, σ_2, and σ_3 are called the principal stresses. The magnitudes of the principal stresses, σ_p, are the roots of

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0, \tag{1.10}$$

where I_1, I_2, and I_3 are called the *invariants* of the stress tensor. They are

$$\begin{aligned} I_1 &= \sigma_{xx} + \sigma_{yy} + \sigma_{zz}, \\ I_2 &= \sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2 - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} - \sigma_{xx}\sigma_{yy}, \quad \text{and} \\ I_3 &= \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{yz}\sigma_{zx}\sigma_{xy} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2. \end{aligned} \tag{1.11}$$

The first invariant $I_1 = -p/3$ where p is the pressure. I_1, I_2, and I_3 are independent of the orientation of the axes. Expressed in terms of the principal stresses, they are

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3, \\ I_2 &= -\sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2, \quad \text{and} \\ I_3 &= \sigma_1\sigma_2\sigma_3. \end{aligned} \tag{1.12}$$

EXAMPLE 1.1: Consider a stress state with $\sigma_x = 70$ MPa, $\sigma_y = 35$ MPa, $\tau_{xy} = 20$, $\sigma_z = \tau_{zx} = \tau_{yz} = 0$. Find the principal stresses using equations 1.10 and 1.11.

SOLUTION: Using equations 1.11, $I_1 = 105$ MPa, $I_2 = -2050$ MPa, $I_3 = 0$. From equation 1.10, $\sigma_p^3 - 105\sigma_p^2 + 2050\sigma_p + 0 = 0$, so

$$\sigma_p^2 - 105\sigma_p + 2{,}050 = 0.$$

The principal stresses are the roots $\sigma_1 = 79.1$ MPa, $\sigma_2 = 25.9$ MPa, and $\sigma_3 = \sigma_z = 0$.

EXAMPLE 1.2: Repeat Example 1.1 with $I_3 = 170{,}700$.

SOLUTION: The principal stresses are the roots of $\sigma_p^3 - 105\sigma_p^2 + 2050\sigma_p + 170{,}700 = 0$. Since one of the roots is $\sigma_z = \sigma_3 = -40$, $\sigma_p + 40 = 0$ can be factored out. This gives $\sigma_p^2 - 105\sigma_p + 2050 = 0$, so the other two principal stresses are $\sigma_1 = 79.1$ MPa, $\sigma_2 = 25.9$ MPa. This shows that when σ_z is one of the principal stresses, the other two principal stresses are independent of σ_z.

1.4 MOHR'S CIRCLE EQUATIONS

In the special cases where two of the three shear stress terms vanish (e.g., $\tau_{yx} = \tau_{zx} = 0$), the stress σ_z normal to the xy plane is a principal stress and the other two principal stresses lie in the xy plane. This is illustrated in Figure 1.5.

For these conditions $\ell_{x'z} = \ell_{y'z} = 0$, $\tau_{yz} = \tau_{zx} = 0$, $\ell_{x'x} = \ell_{y'y} = \cos\phi$, and $\ell_{x'y} = -\ell_{y'x} = \sin\phi$. Substituting these relations into equations 1.9 results in

$$\begin{aligned} \tau_{x'y'} &= \cos\phi\sin\phi(-\sigma_x + \sigma_y) + (\cos^2\phi - \sin^2\phi)\tau_{xy}, \\ \sigma_{x'} &= (\cos^2\phi)\sigma_x + (\sin^2\phi)\sigma_y + 2(\cos\phi\sin\phi)\tau_{xy}, \quad \text{and} \\ \sigma_{y'} &= (\sin^2\phi)\sigma_x + (\cos^2\phi)\sigma_y + 2(\cos\phi\sin\phi)\tau_{xy}. \end{aligned} \tag{1.13}$$

These can be simplified with the trigonometric relations

$$\sin 2\phi = 2\sin\phi\cos\phi \quad \text{and} \quad \cos 2\phi = \cos^2\phi - \sin^2\phi \quad \text{to obtain}$$

$$\tau_{x'y'} = -\sin 2\phi(\sigma_x - \sigma_y)/2 + (\cos 2\phi)\tau_{xy}, \tag{1.14a}$$

$$\sigma_{x'} = (\sigma_x + \sigma_y)/2 + \cos 2\phi(\sigma_x - \sigma_y)/2 + \tau_{xy}\sin 2\phi, \quad \text{and} \tag{1.14b}$$

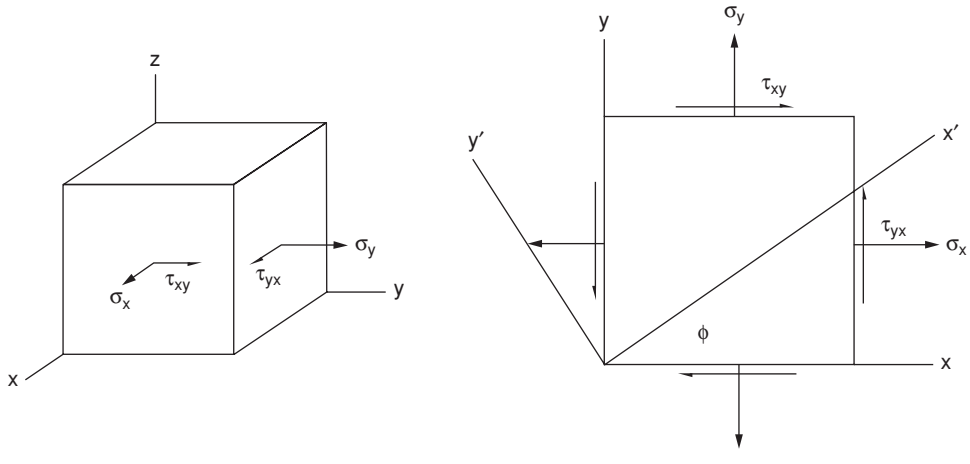
$$\sigma_{y'} = (\sigma_x + \sigma_y)/2 - \cos 2\phi(\sigma_x - \sigma_y)/2 + \tau_{xy}\sin 2\phi. \tag{1.14c}$$

If $\tau_{x'y'}$ is set to zero in equation 1.14a, ϕ becomes the angle θ between the principal axes and the x and y axes. Then

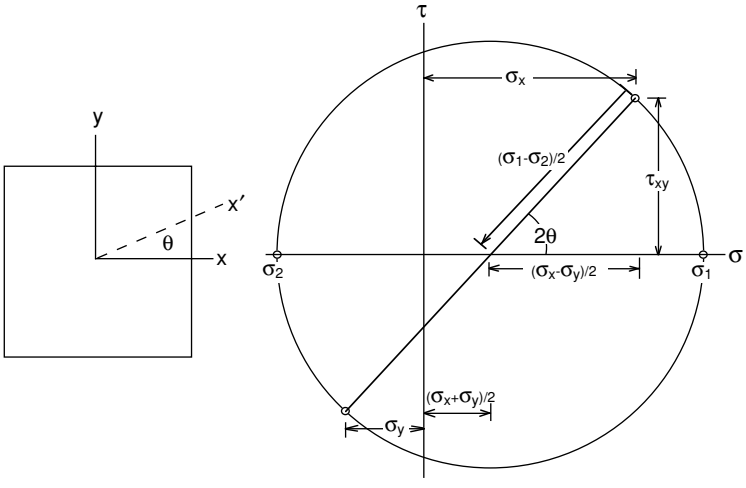
$$\tan 2\theta = \tau_{xy}/[(\sigma_x - \sigma_y)/2]. \tag{1.15}$$

The principal stresses, σ_1 and σ_2, are then the values of $\sigma_{x'}$ and $\sigma_{y'}$

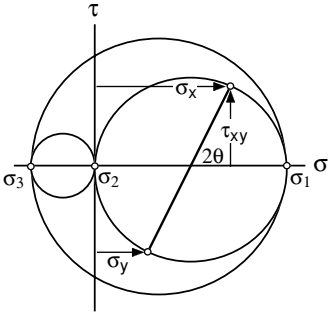
$$\begin{aligned} \sigma_{1,2} &= (\sigma_x + \sigma_y)/2 \pm (1/2)[(\sigma_x - \sigma_y)\cos 2\theta] + \tau_{xy}\sin 2\theta \quad \text{or} \\ \sigma_{1,2} &= (\sigma_x + \sigma_y)/2 \pm (1/2)\left[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2\right]^{1/2}. \end{aligned} \tag{1.16}$$



1.5. Stress state for which the Mohr's circle equations apply.



1.6. Mohr's circle diagram for stress.



1.7. Three-dimensional Mohr's circles for stresses.

A Mohr's* circle diagram is a graphical representation of equations 1.16 and 1.17. They form a circle of radius $(\sigma_1 - \sigma_2)/2$ with the center at $(\sigma_1 + \sigma_2)/2$ as shown in Figure 1.6. The normal stress components are plotted on the ordinate and the shear stress components are plotted on the abscissa.

Using the Pythagorean theorem on the triangle in Figure 1.6,

$$(\sigma_1 - \sigma_2)/2 = \left\{[(\sigma_x - \sigma_y)/2]^2 + \tau_{xy}^2\right\}^{1/2} \tag{1.17}$$

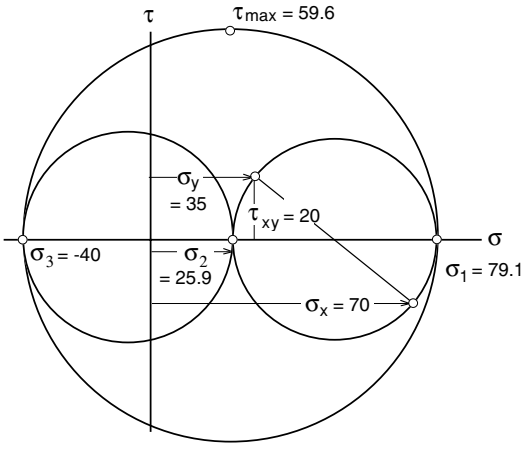
and

$$\tan 2\theta = \tau_{xy}/[(\sigma_x - \sigma_y)/2]. \tag{1.18}$$

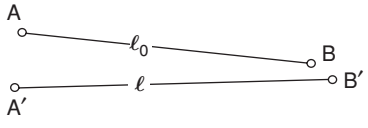
A three-dimensional stress state can be represented by three Mohr's circles as shown in Figure 1.7. The three principal stresses σ_1, σ_2, and σ_3 are plotted on the ordinate. The circles represent the stress state in the 1–2, 2–3, and 3–1 planes.

EXAMPLE 1.3: Construct the Mohr's circle for the stress state in Example 1.2 and determine the largest shear stress.

* O. Mohr, *Zivilingeneur* (1882), p. 113.



1.8. Mohr's circle for stress state in Example 1.2.



1.9. Deformation, translation, and rotation of a line in a material.

SOLUTION: The Mohr's circle is plotted in Figure 1.8. The largest shear stress is $\tau_{max} = (\sigma_1 - \sigma_3)/2 = [79.1 - (-40)]/2 = 59.6\,\text{MPa}$.

1.5 STRAIN

Strain describes the amount of deformation in a body. When a body is deformed, points in that body are displaced. Strain must be defined in such a way that it excludes effects of rotation and translation. Figure 1.9 shows a line in a material that has been that has been deformed. The line has been translated, rotated, and deformed. The deformation is characterized by the *engineering* or *nominal strain*, e:

$$e = (\ell - \ell_0)/\ell_0 = \Delta\ell/\ell_0. \tag{1.19}$$

An alternative definition* is that of *true* or *logarithmic strain*, ε, defined by

$$d\varepsilon = d\ell/\ell, \tag{1.20}$$

which on integrating gives

$$\varepsilon = \ln(\ell/\ell_0) = \ln(1 + e). \tag{1.21}$$

The true and engineering strains are almost equal when they are small. Expressing ε as $\varepsilon = \ln(\ell/\ell_0) = \ln(1 + e)$ and expanding,

$$\varepsilon = e - e^2/2 + e^3/3! - \cdots, \text{ so as } e \to 0, \varepsilon \to e.$$

There are several reasons why true strains are more convenient than engineering strains. The following examples indicate why.

* First defined by P. Ludwig, *Elemente der Technishe Mechanik*, Springer, 1909.

EXAMPLE 1.4:

(a) A bar of length ℓ_0 is uniformly extended until its length $\ell = 2\ell_0$. Compute the values of the engineering and true strains.
(b) To what final length must a bar of length ℓ_0 be compressed if the strains are the same (except sign) as in part (a)?

SOLUTION:

(a) $e = \Delta\ell/\ell_0 = 1.0$, $\varepsilon = \ln(\ell/\ell_0) = \ln 2 = 0.693$.
(b) $e = -1 = (\ell - \ell_0)/\ell_0$, so $\ell = 0$. This is clearly impossible to achieve.

$$\varepsilon = -0.693 = \ln(\ell/\ell_0), \text{ so } \ell = \ell_0 \exp(0.693) = \ell_0/2.$$

EXAMPLE 1.5: A bar 10 cm long is elongated to 20 cm by rolling in three steps: 10 cm to 12 cm, 12 cm to 15 cm, and 15 cm to 20 cm.

(a) Calculate the engineering strain for each step and compare the sum of these with the overall engineering strain.
(b) Repeat for true strains.

SOLUTION:

(a) $e_1 = 2/10 = 0.20$, $e_2 = 3/12 = 0.25$, $e_3 = 5/15 = 0.333$, $e_{\text{tot}} = 0.20 + .25 + .333 = 0.783$. $e_{\text{overall}} = 10/10 = 1$.
(b) $\varepsilon_1 = \ln(12/10) = 0.182$, $\varepsilon_2 = \ln(15/12) = 0.223$, $\varepsilon_3 = \ln(20/15) = 0.288$, $\varepsilon_{\text{tot}} = 0.693$, $\varepsilon_{\text{overall}} = \ln(20/10) = 0.693$.

With true strains, the sum of the increments equals the overall strain, but this is not so with engineering strains.

EXAMPLE 1.6: A block of initial dimensions ℓ_0, w_0, t_0 is deformed to dimensions of ℓ, w, t.

(a) Calculate the volume strain, $\varepsilon_{\text{v}} = \ln(v/v_0)$ in terms of the three normal strains, ε_ℓ, ε_w and ε_t.
(b) Plastic deformation causes no volume change. With no volume change, what is the sum of the three normal strains?

SOLUTION:

(a) $\varepsilon_{\text{v}} = \ln[(\ell wt)/(\ell_0 w_0 t_0)] = \ln(\ell/\ell_0) + \ln(w/w_0) + \ln(t/t_0) = \varepsilon_\ell + \varepsilon_w + \varepsilon_t$.
(b) If $\varepsilon_{\text{v}} = 0$, $\varepsilon_\ell + \varepsilon_w + \varepsilon_t = 0$.

Examples 1.4, 1.5, and 1.6 illustrate why true strains are more convenient than engineering strains.

1. True strains for an equivalent amount of tensile and compressive deformation are equal except for sign.
2. True strains are additive.
3. The volume strain is the sum of the three normal strains.

EXAMPLE 1.7: Calculate the ratio of ε/e for $e = 0.001, 0.01, 0.02, 0.05, 0.1$, and 0.2.

SOLUTION:

For $e = 0.001$, $\varepsilon/e = \ln(1.001)/0.001 = 0.0009995/0.001 = 0.9995$.
For $e = 0.01$, $\varepsilon/e = \ln(1.01)/0.01 = 0.00995/0.01 = 0.995$.
For $e = 0.02$, $\varepsilon/e = \ln(1.02)/0.02 = 0.0198/0.02 = 0.990$.
For $e = 0.05$, $\varepsilon/e = \ln(1.05)/0.05 = 0.04879/0.05 = 0.9758$.
For $e = 0.1$, $\varepsilon/e = \ln(1.1)/0.1 = 0.0953/0.1 = 0.953$.
For $e = 0.2$, $\varepsilon/e = \ln(1.2)/0.2 = 0.1823/0.2 = 0.9116$.

As e gets larger the difference between ε and e become greater.

1.6 SMALL STRAINS

Figure 1.10 shows a small two-dimensional element, $ABCD$, deformed into $A'B'C'D'$ where the displacements are u and v. The normal strain, e_{xx}, is defined as

$$e_{xx} = (A'D' - AD)/AD = A'D'/AD - 1. \tag{1.22}$$

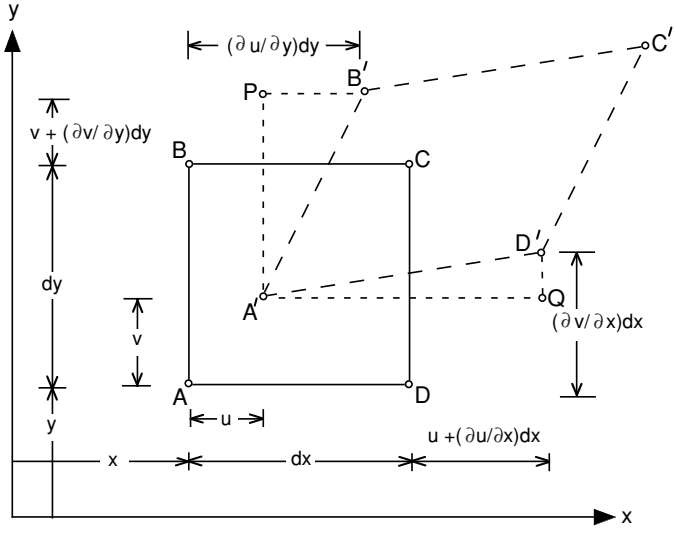
Neglecting the rotation

$$e_{xx} = A'D'/AD - 1 = \frac{dx - u + u + (\partial u/\partial x)\,dx}{dx} - 1 \quad \text{or}$$
$$e_{xx} = \partial u/\partial x. \tag{1.23}$$

Similarly, $e_{yy} = \partial v/\partial y$ and $e_{zz} = \partial w/\partial z$ for a three-dimensional case.

The shear strains are associated with the angles between AD and $A'D'$ and between AB and $A'B'$. For small deformations

$$\angle^{AD}_{A'D'} \approx \partial v/\partial x \quad \text{and} \quad \angle^{AB}_{A'B'} = \partial u/\partial y. \tag{1.24}$$



1.10. Distortion of a two-dimensional element.

The total shear strain is the sum of these two angles,

$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \tag{1.25a}$$

Similarly,

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \text{and} \tag{1.25b}$$

$$\gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}. \tag{1.25c}$$

This definition of shear strain, γ, is equivalent to the simple shear measured in a torsion of shear test.

1.7 THE STRAIN TENSOR

If tensor shear strains ε_{ij} are defined as

$$\varepsilon_{ij} = (1/2)\gamma_{ij}, \tag{1.26}$$

small shear strains form a tensor,

$$\varepsilon_{ij} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{yx} & \varepsilon_{zx} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{zy} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{vmatrix}. \tag{1.27}$$

Because small strains form a tensor, they can be transformed from one set of axes to another in a way identical to the transformation of stresses. Mohr's circle relations can be used. It must be remembered, however, that $\varepsilon_{ij} = \gamma_{ij}/2$ and that the transformations hold only for small strains. If $\gamma_{yz} = \gamma_{zx} = 0$,

$$\varepsilon_{x'} = \varepsilon_x \ell_{x'x}^2 + \varepsilon_y \ell_{x'y}^2 + \gamma_{xy} \ell_{x'x} \ell_{x'y} \tag{1.28}$$

and

$$\gamma_{x'y'} = 2\varepsilon_x \ell_{x'x} \ell_{y'x} + 2\varepsilon_y \ell_{x'y} \ell_{y'y} + \gamma_{xy}(\ell_{x'x} \ell_{y'y} + \ell_{y'x} \ell_{x'y}). \tag{1.29}$$

The principal strains can be found from the Mohr's circle equations for strains,

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm (1/2)\left[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}. \tag{1.30}$$

Strains on other planes are given by

$$\varepsilon_{x,y} = (1/2)(\varepsilon_1 + \varepsilon_2) \pm (1/2)(\varepsilon_1 - \varepsilon_2)\cos 2\theta \tag{1.31}$$

and

$$\gamma_{xy} = (\varepsilon_1 - \varepsilon_2)\sin 2\theta. \tag{1.32}$$

1.8 ISOTROPIC ELASTICITY

Although the thrust of this book is on plastic deformation, a short treatment of elasticity is necessary to understand springback and residual stresses in forming processes.