Asymptotic Analysis of Random Walks

This book focuses on the asymptotic behaviour of the probabilities of large deviations of the trajectories of random walks with 'heavy-tailed' (in particular, regularly varying, sub- and semiexponential) jump distributions. Large deviation probabilities are of great interest in numerous applied areas, typical examples being ruin probabilities in risk theory, error probabilities in mathematical statistics and buffer-overflow probabilities in queueing theory. The classical large deviation theory, developed for distributions decaying exponentially fast (or even faster) at infinity, mostly uses analytical methods. If the fast decay condition fails, which is the case in many important applied problems, then direct probabilistic methods usually prove to be efficient. This monograph presents a unified and systematic exposition of large deviation theory for heavy-tailed random walks. Most of the results presented in the book are appearing in a monograph for the first time. Many of them were obtained by the authors.

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Asymptotic Analysis of Random Walks

Heavy-Tailed Distributions

A.A. BOROVKOV

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Notation

This list includes only the notation used systematically throughout the book.

Random variables and events

$\xi_1, \xi_2 \ldots$ are independent random variables (r.v.'s), assumed to be identically distributed in Chapters 1–11, 16 (in which case $\xi_j \stackrel{d}{=} \xi$)
$\xi(a) = \xi - a$
$\xi^{\langle y \rangle}$ is an r.v. ξ 'truncated' at the level y: $\mathbf{P}(\xi^{\langle y \rangle} < t) = \mathbf{P}(\xi < t)/\mathbf{P}(\xi < y)$, $t \leqslant y$
$\xi^{(\lambda)}$ is an r.v. with distribution $\mathbf{P}(\xi^{(\lambda)} \in dt) = (e^{\lambda t}/\varphi(\lambda))\mathbf{P}(\xi \in dt)$ (the Cramér transform)
$\overline{\xi}_n = \max\limits_{k \leqslant n} \xi_k$
$S_n = \sum_{j=1}^n \xi_j$
$\overline{S}_n = \max\limits_{k \leqslant n} S_k$
$\overline{S} = \sup\limits_{k \geqslant 0} S_k$
$\underline{S}_n = \min\limits_{k \leqslant n} S_k$
$\widehat{S}_n = \max\limits_{k \leqslant n} |S_k| \equiv \max\{\overline{S}_n, \underline{S}_n\}$
$S_n(a) = \sum_{i=1}^n \xi_i(a) \equiv S_n - an$
$\overline{S}_n(a) = \max\limits_{k \leqslant n} S_k(a) = \max\limits_{k \leqslant n}(S_k - ak)$
$\overline{S}(a) = \sup_{k \geqslant 0} S_k(a) = \sup_{k \geqslant 0}(S_k - ak)$
$S_n^{\langle y \rangle} = \sum_{j=1}^n \xi_j^{\langle y \rangle}$
$S_n^{(\lambda)} = \sum_{j=1}^n \xi_j^{(\lambda)}$
$\tau_1, \tau_2, \ldots$ are independent identically distributed r.v.'s ($\tau_j > 0$ in Chapter16)
$\tau \stackrel{d}{=} \tau_j$
$\mathbf{t}_k = \sum_{j=1}^k \tau_j$
G_n is one of the events $\{S_n \geqslant x\}$, $\{\overline{S}_n \geqslant x\}$ or $\{\max_{k \leqslant n}(S_k - g(k)) \geqslant 0\}$
$\mathbb{G}_T = \{\sup_{t \leqslant T}(S(t) - g(t)) \geqslant 0\}$ (in Chapters 15 and 16)
$\mathbf{1}(A)$ is the indicator of the event A

$\stackrel{d}{=}$, $\stackrel{d}{\leqslant}$, $\stackrel{d}{\geqslant}$ are equality and inequalities between r.v.'s in distribution
$\Rightarrow$ is used to denote convergence of r.v.'s in distribution

Distributions and their characteristics

The notation $\zeta \Subset \mathbf{G}$ means that the r.v. ζ has the distribution $\mathbf{G}$
The notation $\zeta_n \Longleftrightarrow \mathbf{G}$ means that the distributions of the r.v.'s ζ_n converge weakly to the distribution $\mathbf{G}$ (as $n \to \infty$)
$\mathbf{F}_j$ is the distribution of ξ_j ($\mathbf{F}_j = \mathbf{F}$ in Chapters 1–11, 16)
$F_+(t) = \mathbf{F}([t,\infty)) \equiv \mathbf{P}(\xi \geqslant t), \quad F_{j,+}(t) = \mathbf{F}_j([t,\infty))$
$F_-(t) = \mathbf{F}((-\infty,-t)) \equiv \mathbf{P}(\xi < -t), \quad F_{j,-}(t) = \mathbf{F}_j((-\infty,-t))$
$F(t) = F_-(t) + F_+(t), \quad F_j(t) = F_{j,-}(t) + F_{j,+}(t)$
$F_I(t) = \int_0^t F(u)\,du, \quad F^I(t) = \int_t^\infty F(u)\,du$
$V(t), W(t), U(t)$ are regularly varying functions (r.v.f.'s) (in Chapters 1–4):
$V(t) = t^{-\alpha}L(t)$, $\alpha > 0$
$W(t) = t^{-\beta}L_W(t)$, $\beta > 0$
$U(t) = t^{-\gamma}L_U(t)$, $\gamma > 0$
$L(t), L_W(t), L_U(t), L_Y(t)$ are slowly varying functions (s.v.f.'s), corresponding to V, W, U, Y
$V(t) = e^{-l(t)}$, $l(t) = t^\alpha L(t)$, $\alpha \in (0,1)$, $L(t)$ is an s.v.f. (in Chapter 5)
$l(t) = t^\alpha L(t)$ is the exponent of a semiexponential distribution
$\widehat{V}(t) = \max\{V(t), W(t)\}$
$\mathbf{F}_\tau$ is the distribution of τ
$\mathbf{G}$ is the distribution of ζ
α, β are the exponents of the right and left regularly varying distribution tails of ξ respectively, or those of their regularly varying majorants (or minorants)
$\hat{\alpha} = \max\{\alpha, \beta\}$
$\check{\alpha} = \min\{\alpha, \beta\}$
$d = \operatorname{Var}\xi = \mathbf{E}(\xi - \mathbf{E}\xi)^2$
$\mathfrak{f}(\lambda) = \mathbf{E}e^{i\lambda\xi}$ is the characteristic function (ch.f.) of ξ
$\mathfrak{g}(\lambda) = \mathbf{E}e^{i\lambda\zeta}$ is the ch.f. of ζ
$\varphi(\lambda) = \mathbf{E}e^{\lambda\xi}$ is the moment generating function of ξ
(α, ρ) are the parameters of the limiting stable law
$\mathbf{F}_{\alpha,\rho}$ is the (standard) stable distribution with parameters (α, ρ)
$F_{\alpha,\rho,+}(t) = \mathbf{F}_{\alpha,\rho}([t,\infty)), \quad F_{\alpha,\rho,-} = \mathbf{F}_{\alpha,\rho}((-\infty,-t)), \quad t > 0$
$F_{\alpha,\rho}(t) = F_{\alpha,\rho,+}(t) + F_{\alpha,\rho,-}(t), \quad t > 0$
$\boldsymbol{\Phi}$ is the standard normal distribution
$\Phi(t)$ is the standard normal distribution function

Conditions on distributions

$[\,\cdot\,,=] \quad \Leftrightarrow \quad F_+(t) = V(t), \quad t > 0$
$[\,\cdot\,,<] \quad \Leftrightarrow \quad F_+(t) \leqslant V(t), \quad t > 0$

$[\,\cdot\,,>]\quad\Leftrightarrow\quad F_+(t)\geqslant V(t),\quad t>0$
$[=,\,\cdot\,]\quad\Leftrightarrow\quad F_-(t)=W(t),\quad t>0$
$[<,\,\cdot\,]\quad\Leftrightarrow\quad F_-(t)\leqslant W(t),\quad t>0$
$[>,\,\cdot\,]\quad\Leftrightarrow\quad F_-(t)\geqslant W(t),\quad t>0$
$[=,=]\quad\Leftrightarrow\quad F_+(t)=V(t),\ F_-(t)=W(t),\quad t>0$
$[<,<]\quad\Leftrightarrow\quad F_+(t)\leqslant V(t),\ F_-(t)\leqslant W(t),\quad t>0$
$[>,>]\quad\Leftrightarrow\quad F_+(t)\geqslant V(t),\ F_-(t)\geqslant W(t),\quad t>0$
$[\mathbf{R}_{\alpha,\rho}]$ means that $F(t)=t^{-\alpha}L_F(t)$, $\alpha\in(0,2]$, where $L_F(t)$ is an s.v.f. and there exists the limit

$$\lim_{t\to\infty}\frac{F_+(t)}{F(t)}=:\rho_+=\frac{1}{2}(\rho+1)\in[0,1]$$

$[\mathbf{D}_{(h,q)}]$, $h\in(0,2]$, are conditions on the smoothness of $F(t)$ at infinity; see § 3.4
$[\mathbf{D}_{(k,q)}]$, $k=1,2,\dots$, are generalized conditions of the differentiability of $F(t)$ at infinity; see § 4.4
$[<]$ means that $F_\tau(t)\leqslant V_\tau(t):=t^{-\gamma}L_\tau(t)$, where L_τ is an s.v.f.

Scalings

$$b(n)=\begin{cases}F^{(-1)}(1/n) & \text{if}\quad \mathbf{E}\xi^2=\infty,\ \alpha<2,\\ Y^{(-1)}(1/n) & \text{if}\quad \mathbf{E}\xi^2=\infty,\ \alpha=2,\\ \sqrt{nd} & \text{if}\quad d=\operatorname{Var}\xi<\infty\end{cases}$$

$$\sigma(n)=\begin{cases}V^{(-1)}(1/n) & \text{if}\quad \mathbf{E}\xi^2=\infty,\\ \sqrt{(\alpha-2)dn\ln n} & \text{if}\quad d=\operatorname{Var}\xi<\infty\end{cases}$$

$\sigma_W(n)=W^{(-1)}(1/n)$
$\sigma_1(n)=w_1^{(-1)}(1/n)$, where $w_1(t)=t^{-2}l(t)$ (in Chapter5)
$\sigma_2(n)=w_2^{(-1)}(1/n)$, where $w_2(t)=t^{-2}l^2(t)$ (in Chapter5)

Combined conditions

$[\mathbf{Q}_1]$: $\mathbf{E}\xi^2=\infty$, $[<,<]$, $W(t)\leqslant cV(t)$, $x\to\infty$ and $nV(x)\to 0$
$[\mathbf{Q}_2]$: $\mathbf{E}\xi^2=\infty$, $[<,<]$, $x\to\infty$ and $n\widehat{V}\left(\dfrac{x}{\ln x}\right)<c<\infty$
$[\mathbf{Q}]=[\mathbf{Q}_1]\cup[\mathbf{Q}_2]$

Distribution classes

$\mathcal{L}$ is the class of distributions with asymptotically locally constant tails (or of their distribution tails)
$\mathcal{R}$ is the class of distributions with right tails regularly varying at infinity (or the class of their tails); in Chapters 2–4, the condition $F_+\in\mathcal{R}$ coincides with condition $[\,\cdot\,,=]$
$\mathcal{ER}$ is the class of regularly varying exponentially decaying distributions (or of their tails)
$\mathcal{S}e$ is the class of semiexponential distributions (or of their tails); in Chapter 5, the condition $F_+\in\mathcal{S}e$ coincides with condition $[\,\cdot\,,=]$

$\mathcal{S}_+$ is the class of subexponential distributions on $\mathbb{R}_+$ (or of their tails)
$\mathcal{S}$ is the class of subexponential distributions (or of their tails)
$\mathcal{S}_{\mathrm{loc}}$ is the class of locally subexponential distributions
$\mathcal{C}$ is the class of distributions satisfying the Cramér condition
$\mathcal{M}_s$ is the class of distributions satisfying the condition $\mathbf{E}|\xi|^s < \infty$
$\mathfrak{S}$ is the class of stable distributions
$\mathcal{N}_{\alpha,\rho}$ is the domain of normal attraction to the stable law $\mathbf{F}_{\alpha,\rho}$

Miscellaneous

$\sim$ is the relation of asymptotic equivalence: $A \sim B \Leftrightarrow A/B \to 1$
$\asymp$ is the relation of asymptotic comparability: $A \asymp B \Leftrightarrow A = O(B),\ B = O(A)$
$x^+ = \max\{0, x\}$
$\lfloor x \rfloor$ denotes the integral part of x
$r = x/y \geqslant 1$
$\phi = \operatorname{sign} \lambda$
$\Pi = \Pi(x, n) = nV(x)$
$\Pi(y) = \Pi(y, n)$
$\widehat{\Pi}(y) = n\widehat{V}(x)$
$\Delta[x) = [x, x + \Delta),\ \ \Delta > 0$ (in Chapter 9, $\Delta[x)$ is a cube in $\mathbb{R}^d$ with edge length Δ)
$(x, y) = \sum_{i=1}^{d} x^{(i)} y^{(i)}$ is the scalar product of the vectors $x, y \in \mathbb{R}^d$
$|x| = (x, x)^{1/2}$
$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere in $\mathbb{R}^d$
$\mathrm{e}(x) = x/|x|\,,\ x \in \mathbb{R}^d$

Introduction

This book is concerned with the asymptotic behaviour of the probabilities of rare events related to large deviations of the trajectories of random walks, whose jump distributions decay not too fast at infinity and possess some form of 'regular behaviour'. Typically, we will be considering regularly varying, subexponential, semiexponential and other similar distributions. For brevity, all these distributions will be referred to in this Introduction as 'regular'. As the main key words for the contents of the present book one could list the following: *random walks, large deviations, slowly decaying* and, in particular, *regular distributions*. To the question why the above-mentioned themes have become the subject of a separate monograph, an answer relating to each of these key concepts can be given.

• **Random walks** form a classical object of probability theory, the study of which is of tremendous theoretical interest. They constitute a mathematical model of great importance for applications in mathematical statistics (sequential analysis), risk theory, queueing theory and so on.

• **Large deviations and rare events** are of great interest in all these applied areas, since computing the asymptotics of large deviation probabilities enables one to find, for example, small error probabilities in mathematical statistics, small ruin probabilities in risk theory, small buffer overflow probabilities in queueing theory and so on.

• **Slowly decaying** and, in particular, **regular distributions** present, when one is studying large deviation probabilities, an alternative to distributions decaying exponentially fast at infinity (for which Cramér's condition holds; the meaning of the terms we are using here will be explained in more detail in what follows). The first classical results in large deviation theory were obtained for the case of distributions decaying exponentially fast. However, this condition of fast (exponential) decay fails in many applied problems. Thus, for instance, the 'empirical distribution tails' for insurance claim sizes, for the sizes of files sent over the Internet and also for many other data sets as well, usually decay as a power function (see e.g. [1]). However, in the problems of, say, mathematical statistics, the assumption of fast decay of the distributions is often adequate as it reflects the nature of the problem. Therefore, both classes of distributions, regular and fast decaying, are of great interest.

Random walks with fast decaying distributions will be considered in a separate book (a companion volume to the present monograph). The reason for this is that studying random walks with regular distributions requires a completely different approach in comparison with the case of fast decaying distributions, since for regular distributions the large deviation probabilities are mostly formed by contributions from the distribution *tails* (i.e. on account of the large jumps in the random walk trajectory), while for distributions decaying exponentially fast they are formed by contributions from the *'central parts'* of the distributions. In the latter case analytical methods prove to be efficient, and everything is determined by the behaviour of the Laplace transforms of the jump distributions. In the former case, direct probabilistic approaches prove to be more efficient. However, until now the results for regular distributions were of a disconnected character, and referred to different special cases. The writing of the present monograph was undertaken as an attempt to present a unified exposition of the theory on the basis of a common approach; a large number of the results we present are new.

Before surveying the contents of the book, we will make a few further, more detailed, remarks of a general character in order to make the naturalness of the objects of study, and also the logic and structure of the monograph, clearer to the reader.

The application of probability theory as a mathematical discipline is most efficient when one is studying phenomena where a large number of random factors are present. The influence of such factors is, as a rule, additive (or can be reduced to such), especially in cases where the individual contribution of each factor is small. Hence sums of random variables have always been (and will be) among the main objects of research in probability theory. A vast literature is devoted to the study of the main asymptotic laws describing the distributions of sums of large numbers of random summands (see e.g. [130, 152, 186, 223, 121, 122]). The most advanced results in this direction were obtained for sums of *independent identically distributed* (i.i.d.) random variables (r.v.'s).

Let $\xi, \xi_1, \xi_2, \ldots$ be i.i.d. (possibly vector-valued) r.v.'s. Put $S_0 := 0$ and

$$S_n := \sum_{i=1}^{n} \xi_i, \qquad n = 1, 2, \ldots$$

The sequence $\{S_n;\, n \geqslant 1\}$ is called a *random walk*. The following assertions constitute the fundamental classical limit theorems for random walks.

1. The strong law of large numbers. If there exists a finite expectation $\mathbf{E}\xi$ then, as $n \to \infty$,

$$\frac{S_n}{n} \to \mathbf{E}\xi \qquad \text{almost surely (a.s.).} \tag{0.0.1}$$

One could call the value $n\mathbf{E}\xi$ the *first-order approximation* to the sum S_n.

2. The central limit theorem. If $\mathbf{E}\xi^2 < \infty$ then, as $n \to \infty$,

$$\zeta_n := \frac{S_n - n\mathbf{E}\xi}{\sqrt{nd}} \Rightarrow \zeta \Subset \mathbf{\Phi}, \tag{0.0.2}$$

where $d = \operatorname{Var}\xi = \mathbf{E}\xi^2 - (\mathbf{E}\xi)^2$ is the variance of the r.v. ξ, the symbol $\Rightarrow$ denotes the (weak) convergence of the r.v.'s in distribution and the notation $\zeta \Subset \mathbf{F}$ says that the r.v. ζ has the distribution $\mathbf{F}$; in this case, $\mathbf{F} = \mathbf{\Phi}$ is the standard normal distribution (with parameters (0,1)). One could call the quantity $n\mathbf{E}\xi + \zeta\sqrt{nd}$ the *second-order approximation* to S_n.

Since the limiting distribution $\mathbf{\Phi}$ in (0.0.2) is continuous, the relation (0.0.2) is equivalent to the following one: for any $v \in \mathbb{R}$ we have

$$\mathbf{P}(\zeta_n \geqslant v) \to \mathbf{P}(\zeta \geqslant v) \quad \text{as} \quad n \to \infty,$$

and, moreover, this convergence is uniform in v. In other words, for deviations of the form $x = n\mathbf{E}\xi + v\sqrt{nd}$,

$$\mathbf{P}(S_n \geqslant x) \sim \mathbf{P}\big(\zeta \geqslant (x - n\mathbf{E}\xi)/\sqrt{nd}\big) = 1 - \Phi(v) \quad \text{as} \quad n \to \infty \tag{0.0.3}$$

uniformly in $v \in [v_1, v_2]$, where $-\infty < v_1 \leqslant v_2 < \infty$ are fixed numbers and Φ is the standard normal distribution function (here and in what follows, the notation $A \sim B$ means that $A/B \to 1$ under the indicated passage to the limit).

3. Convergence to stable laws. If the expectation of the r.v. ξ is infinite or does not exist, then the first-order approximation for the sum S_n can only be found when the sum of the *right and left tails* of the distribution of ξ, i.e. the function

$$F(t) := \mathbf{P}(\xi \geqslant t) + \mathbf{P}(\xi < -t), \qquad t > 0,$$

is regularly varying as $t \to \infty$; it can be represented as

$$F(t) = t^{-\alpha} L(t), \tag{0.0.4}$$

where $\alpha \in (0, 1]$ and $L(t)$ is a slowly varying function (s.v.f.) as $t \to \infty$. The same can be said about the *second-order approximation* for S_n in the case when $\mathbf{E}|\xi| < \infty$ but $\mathbf{E}\xi^2 = \infty$. In this case, we have $\alpha \in [1, 2]$ in (0.0.4).

For these two cases, we have the following assertion. For simplicity's sake, assume that $\alpha < 2$, $\alpha \neq 1$; we will also assume that $\mathbf{E}\xi = 0$ when the expectation is finite (the 'boundary case' $\alpha = 1$ is excluded to avoid the necessity of non-trivial centring of the sums S_n when $\mathbf{E}\xi = \pm\infty$ or the expectation does not exist).

Let $F_+(t) := \mathbf{P}(\xi \geqslant t)$, *let* (0.0.4) *hold and let there exist the limit*

$$\lim_{t\to\infty} \frac{F_+(t)}{F(t)} =: \rho_+ \in [0, 1]$$

(if $\rho_+ = 0$ then the right tail of the distribution does not need to be regularly varying). Denote by

$$F^{(-1)}(x) := \inf\{t > 0 :\ F(t) \leqslant x\}, \qquad x > 0,$$

the (generalized) inverse function for F, and put

$$b(n) := F^{(-1)}(n^{-1}) = n^{1/\alpha}L_1(n),$$

where L_1 is also be an s.v.f. (see § 1.1). Then, as $n \to \infty$,

$$\frac{S_n}{b(n)} \Rightarrow \zeta^{(\alpha,\rho)} \in \mathbf{F}_{\alpha,\rho}, \tag{0.0.5}$$

where $\mathbf{F}_{\alpha,\rho}$ is the standard stable law with parameters α and $\rho = 2\rho_+ - 1$.

For completeness of exposition, we will present a formal proof of the above assertion for all $\alpha \in (0, 2]$ in § 1.5.

The relation (0.0.5), similarly to (0.0.3), means that, for $x = vb(n)$,

$$\mathbf{P}(S_n \geqslant x) \sim 1 - F_{\alpha,\rho,+}(v) \quad \text{as} \quad n \to \infty \tag{0.0.6}$$

uniformly in $v \in [v_1, v_2]$ for fixed $v_1 \leqslant v_2$ from $(0, \infty)$, where $F_{\alpha,\rho,+}(v) = \mathbf{F}_{\alpha,\rho}([v,\infty))$ is the right tail of the distribution $\mathbf{F}_{\alpha,\rho}$.

The assertions (0.0.2), (0.0.3) and (0.0.5), (0.0.6) give a satisfactory answer for the behaviour of the probabilities of the form $\mathbf{P}(S_n \geqslant x)$ for large n only for deviations of the form $x = vb(n)$, where v is moderately large and $b(n)$ is the scaling factor in the respective limit theorem ($b(n) = \sqrt{nd}$ in the case $\mathbf{E}\xi^2 < \infty$, when we also assume that $\mathbf{E}\xi = 0$). For example, in the finite variance case it is recommended to deal very carefully with the normal approximation values given by (0.0.3) for $v > 3$ and moderately large values of n (say, $n \leqslant 100$). This leads to a natural setting for the problem on the asymptotic behaviour of the probabilities $\mathbf{P}(S_n \geqslant x)$ for $x \gg b(n)$, all the more so since, as we have already noted, it is precisely such 'large deviations' that are often of interest in applied problems. Such probabilities are referred to as the *probabilities of large deviations of the sums S_n*.

So far we have only discussed questions related to the distributions of the partial sums S_n. However, in many applications (in particular, as already mentioned, in mathematical statistics, queueing theory, risk theory and other areas) questions related to the behaviour of the *entire trajectory* $S_1, S_2, \ldots, S_n$ are of no less importance. Thus, of interest is the problem of computing the probability

$$\mathbf{P}\big(g_1(k) < S_k < g_2(k);\ k = 1, \ldots, n\big) \tag{0.0.7}$$

for two given sequences ('boundaries') $g_1(k)$ and $g_2(k)$, $k = 1, 2, \ldots,$ or the probability of the complementary event that the trajectory $\{S_k;\ k = 1, \ldots, n\}$ will leave at least once the corridor specified by the functions $g_i(k)$. These are the so-called *boundary problems* for random walks. The simplest is the problem on the limiting distribution of the variables

$$\overline{S}_n := \max_{k\leqslant n} S_k \qquad \text{and} \qquad \overline{S}_n(a) := \max_{k\leqslant n}(S_k - ak)$$

for a given constant a.

The following is known about the asymptotics of the probability (0.0.7). Let $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$ (and without loss of generality, one can assume that in this case $\mathbf{E}\xi^2 = 1$), and let the boundaries g_i be given by the relations

$$g_i(k) = \sqrt{n} f_i\big(k/n\big), \qquad i = 1, 2, \tag{0.0.8}$$

where $f_1 < f_2$ are some fixed sufficiently regular (e.g. piecewise smooth) functions on $[0, 1]$, $f_1(0) < 0 < f_2(0)$. Then by the Kolmogorov-Petrovskii theorem (see e.g. [162]) the probability (0.0.7) converges as $n \to \infty$ to the value $P(0, 0)$ of the solution $P(t, z)$ to the boundary problem for the parabolic equation

$$\frac{\partial P}{\partial t} + \frac{1}{2}\frac{\partial^2 P}{\partial z^2} = 0$$

in the region $\big\{(t, z) : 0 < t < 1,\ f_1(t) < z < f_2(t)\big\}$ with boundary conditions

$$\begin{cases} P(t, f_i(t)) = 0, & t \in [0, 1], \quad i = 1, 2, \\ P(1, z) = 1, & z \in (f_1(1), f_2(1)). \end{cases}$$

The above assertion also follows from the so-called *invariance principle* (also known as the functional central limit theorem). According to the principle, the distribution of the random polygon $\{\zeta_n(t);\ t \in [0, 1]\}$ with vertices at the points $(k/n, S_k/\sqrt{n}\,)$, $k = 0, 1, \ldots, n$, in the space $C(0, 1)$ of continuous functions on $[0, 1]$ (endowed with the σ-algebra of Borel sets generated by the uniform distance in $C(0, 1)$) converges weakly to the distribution of the standard Wiener process $\{w(t);\ t \in [0, 1]\}$ in the space $C(0, 1)$ as $n \to \infty$. From this fact it follows that the probability (0.0.7) for the boundaries (0.0.8) converges to the quantity

$$\mathbf{P}\big(f_1(t) < w(t) < f_2(t);\ t \in [0, 1]\big),$$

which, in turn, is given by the solution to the above-mentioned boundary problem for the parabolic equation at the point $(0, 0)$.

A similar 'invariance principle' holds in the case of convergence to stable laws, when $\mathbf{E}\xi^2 = \infty$. In the case where $g_i(k) = b(n)f_i(k/n)$, the probability (0.0.7) converges to the value

$$\mathbf{P}\big(f_1(t) < \zeta(t) < f_2(t);\ t \in [0, 1]\big),$$

where $\zeta(\cdot)$ is the corresponding stable process (the increments of the process $\zeta(\cdot)$ on disjoint time intervals are independent of each other and have, up to a scaling transform, the distribution $\mathbf{F}_{\alpha,\rho}$).

Here we encounter the same problem: the above results do not give a satisfactory answer to the question of the behaviour of probabilities of the form (0.0.7) (or the probabilities of the complementary events) in the case where $g_1(k)$ and $-g_2(k)$ are *large* in comparison with $b(n)$. We again arrive at the large deviation problem but now in the context of boundary problems for random walks. One of the simplest examples here is the problem on the asymptotic behaviour of the probability

$$\mathbf{P}\big(\overline{S}_n(a) \geqslant x\big) \to 0$$

when, say,

$$\mathbf{E}\xi = 0, \qquad a = 0, \qquad x \gg b(n),$$

whereas the case

$$\mathbf{E}\xi = 0, \qquad a > 0, \qquad x \to \infty$$

provides one with another example of a problem on large deviation probabilities; this, however, does not quite fit the above scheme.

The above-mentioned problems on large deviations, along with a number of other related problems, constitute the main object of study in the present book.

Here we should make the following important remark. Even while studying large deviation probabilities for S_n in the case $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$, it turns out that the nature of the asymptotics of $\mathbf{P}(S_n \geqslant x)$ when $x \gg \sqrt{n \ln n}$, and the methods used to find it, strongly depend on the behaviour of the tail $\mathbf{P}(\xi \geqslant t)$ as $t \to \infty$. If the tail vanishes exponentially fast, i.e. the so-called *Cramér condition* is met:

$$\varphi(\lambda) := \mathbf{E}e^{\lambda\xi} < \infty \tag{0.0.9}$$

for some $\lambda > 0$, then, as we have already noted, the asymptotics in question will be formed, roughly speaking, in equal degrees by contributions from all the jumps in the trajectory of the random walk. In this case, the asymptotics are described by laws that are established mainly via analytical calculations and are determined in a wide zone of deviations by the analytic properties of the moment generating function $\varphi(\lambda)$.

If Cramér's condition does not hold then, as when studying the conditions for convergence to stable laws in (0.0.5), we have to assume the regular character of the tails $F_+(t) = \mathbf{P}(\xi \geqslant t)$. Such an assumption can be either a condition of the form (0.0.4) or the condition

$$\mathbf{P}(\xi \geqslant t) = \exp\{-t^{-\alpha}L(t)\}, \qquad \alpha \in (0,1), \tag{0.0.10}$$

where $L(t)$ is an s.v.f. as $t \to \infty$ possessing some smoothness properties. The class of the tails of the form (0.0.4) will be called the class of *regularly varying tails* (*distributions*), and the class specified by the condition (0.0.10) (under some additional assumptions on the function L) will be called the class of *semiexponential tails* (*distributions*).

In the cases (0.0.4) and (0.0.10), the asymptotics of $\mathbf{P}(S_n \geqslant x)$ in situations where x grows fast enough as $n \to \infty$ will, as a rule, be governed by a single large jump. The methods of deriving the asymptotics, as well as the asymptotics itself, turn out to be substantially different from those in the case where Cramér's condition (0.0.9) is met. The methods used here are mostly based on direct probabilistic approaches.

The main objects of study in the present monograph are problems on large deviations for random walks with jumps following regular distributions (in particular, regularly varying, (0.0.4)***, and semiexponential,*** (0.0.10)***, ones).***

There is a great deal of literature on studying these problems. This is especially so for the asymptotics of the distributions $\mathbf{P}(S_n \geqslant x)$ of sums of r.v.'s (see below and also the bibliographic notes to Chapters 2–5). However, the results obtained in this direction so far have been, for the most part, disconnected and incomplete.

Along with conditions of the form (0.0.4) and (0.0.10), one often encounters in the literature the so-called *subexponentiality* property of the right distribution tails, which is characterized by the following relation: for independent copies ξ_1 and ξ_2 of the r.v. ξ one has

$$\mathbf{P}(\xi_1^+ + \xi_2^+ \geqslant t) \sim 2\mathbf{P}(\xi^+ \geqslant t) \quad \text{as} \quad t \to \infty, \tag{0.0.11}$$

where $x^+ = \max\{0, x\}$ is the positive part of x. Distributions from both classes (0.0.4) and (0.0.10) possess this property. Roughly speaking, the classes (0.0.4) and (0.0.10) form a 'regular' part of the class of subexponential distributions. In this connection, it is important to note that the methods of study and the form of the asymptotics of interest for the classes (0.0.4) and (0.0.10) prove in many situations to be substantially different. That is why in this book we will study these classes *separately*, believing that this approach is methodologically well justified.

A short history of the problem. Research in the area of large deviations for random walks with heavy-tailed jumps began in the second half of the twentieth century. At first the main effort was, of course, concentrated on studying the large deviations of the *sums* S_n of r.v.'s. Here one should first of all mention the papers by C. Heyde [141, 145], S.V. Nagaev [201, 206], A.V. Nagaev [194, 195], D.H. Fuk and S.V. Nagaev [127], L.V. Rozovskii [237, 238] and others. These established the basic principle by which the asymptotics of $\mathbf{P}(S_n \geqslant x)$ are formed: the main contribution to the probability of interest comes from trajectories that contain one large jump.

Later on papers began appearing in which this principle was used to find the distribution of the maximum $\overline{S}_n$ of partial sums and also to solve more general boundary problems for random walks (I.F. Pinelis [225], V.V. Godovanchuk [131], A.A. Borovkov [40, 42]).

Somewhat aside from this were papers devoted to the probabilities of large deviations of the maximum of a random walk with negative drift. The first general results were obtained by A.A. Borovkov in [42], while more complete versions (for subexponential summands) were established by N. Veraverbeke [275] and D.A. Korshunov [178].

The authors of the present book began a systematic study of large deviations for random walks with regularly distributed jumps in their papers [51] and [63, 66]. Then the papers [52, 64, 54, 59, 60] and some others appeared, in which the derived results were extended to semiexponential and regular exponentially decaying distributions, to multivariate random walks, to the case of non-identically distributed summands and so on. As a result, a whole range of interesting problems arose, unified by the general approach to their solution and a system of interconnected and rather advanced results that were, as a rule, quite close to unimprovable. As these problems and results were, moreover, of a considerable interest for applications, the idea of writing a monograph on all this became quite natural. The same applies to the related monograph, to be devoted to random walks with fast decaying jump distributions.

More detailed bibliographic references will be given within the exposition in each chapter, and also in the bibliographic notes at the end of the book.

Now we will outline the contents of the book. Chapter 1 contains preliminary results and information that will be used in the sequel. In § 1.1[1] we present the basic properties of slowly varying and regularly varying functions and in § 1.2 the classes of subexponential and semiexponential distributions are introduced. We give conditions characterizing these classes and establish a connection between them. The asymptotic properties of the so-called *functions of* (subexponential) *distributions* are studied in § 1.4. The above-mentioned §§ 1.1–1.4 constitute the first part of the chapter.

In the second part of Chapter 1 (§§ 1.5, 1.6) we present known fundamental limit theorems of probability theory (already briefly mentioned above). Section 1.5 contains a proof of the theorem on the convergence in distribution of the normed sums of i.i.d. r.v.'s to a stable law. A special feature of the proof is the use of an explicit form of the scaling sequence, which enables one to characterize the moderate deviation and large deviation zones using the same terms as those to be employed in Chapters 2 and 3. In § 1.6 we present functional limit theorems (the invariance principle) and the law of the iterated logarithm.

Chapters 2–5 are similar in their contents and structure to each other, and it is appropriate to review them as a whole. They are devoted to studying large deviation probabilities for random walks whose jump distributions belong to one of the following four distribution classes:

(1) regularly varying distributions (or distributions admitting regularly varying majorants or minorants) having no finite means (Chapter 2);
(2) distributions of the above type having finite means but infinite variances (Chapter 3);

[1] The first part of the section number stands for the chapter number.

(3) distributions of the above type having finite variances (Chapter 4);
(4) semiexponential distributions (Chapter 5).

The first sections in all these chapters are devoted to bounding from above the probabilities $\mathbf{P}(\overline{S}_n \geqslant x)$ and $\mathbf{P}(\overline{S}_n(a) \geqslant x)$. The main approach to obtaining these bounds is the same in all four chapters (it is presented in § 2.1), but the results are different depending on the conditions imposed on the majorants of the distribution tails of the r.v. ξ. The same can be said about the lower bounds. The derived two-sided bounds prove to be sharp enough to obtain, under the conditions of Chapters 2–4 in the case when the tails $\mathbf{P}(\xi \geqslant t) = V(t)$ are regularly varying, the asymptotics

$$\mathbf{P}(S_n \geqslant x) \sim nV(x), \qquad \mathbf{P}(\overline{S}_n \geqslant x) \sim nV(x)$$

either in the widest possible zones of deviations x or in zones quite close to the latter.

Each of these four chapters contains sections where, using more precise approaches, we establish the asymptotics of the probabilities

$$\mathbf{P}(S_n \geqslant x), \qquad \mathbf{P}(\overline{S}_n \geqslant x), \qquad \mathbf{P}\big(\overline{S}_n(a) \geqslant x\big)$$

and, moreover, asymptotic expansions for them as well (under additional conditions on the tails $F_+(t)$). An exception is Chapter 2, as under its assumptions the problems of deriving asymptotic expansions and studying the asymptotics of $\mathbf{P}(\overline{S}_n(a) \geqslant x)$ are not meaningful.

Furthermore,

- Chapter 2 contains a finiteness criterion for the supremum of cumulative sums (§ 2.5) and the asymptotics of the renewal function (§ 2.6).
- In Chapters 3 and 4 we obtain integro-local theorems on large deviations of S_n (§§ 3.7, 4.7).
- In Chapter 3 we find conditions for uniform relative convergence to a stable law on the entire axis and establish analogues to the law of the iterated logarithm in the case $\mathbf{E}\xi^2 = \infty$ (§ 3.9).
- In Chapters 3 and 4 we find the asymptotics of the probability $\mathbf{P}\big(\max_{k\leqslant n}(S_k - g(k)) \geqslant 0\big)$ of the crossing of an arbitrary boundary $\{g(k);\, k \geqslant 1\}$ prior to the time n by the random walk (§§ 3.6, 4.6).
- In Chapter 4 we consider the possibility of extending the results on the asymptotic behaviour of $\mathbf{P}(S_n \geqslant x)$ and $\mathbf{P}(\overline{S}_n \geqslant x)$ to wider classes of jump distributions (§ 4.8) and we describe the limiting behaviour of the trajectory $\{S_k;\, k = 1, \ldots, n\}$ given that $S_n \geqslant x$ or $\overline{S}_n \geqslant x$ (§ 4.9).

Chapters 2–5 are devoted to 'heavy-tailed' random walks, i.e. to situations when the jump distributions vanish at infinity more slowly than an exponential function, and indeed this is the main focus of the book.

In Chapter 6, however, we present the main approach to studying the large deviation probabilities for 'light-tailed' random walks – this is the case when the jump distributions vanish at infinity exponentially fast or even faster (i.e. they satisfy Cramér's condition (0.0.9) for some $\lambda > 0$). This is done for the sake of completeness of exposition, and also to ascertain that, in a number of cases, studying 'light-tailed' random walks *can be reduced* to the respective problems for heavy-tailed distributions considered in Chapters 2–5.

In § 6.1 we describe the main method for studying large deviation probabilities when Cramér's condition holds (the method is based on the Cramér transform and the integro-local Gnedenko–Stone–Shepp theorems) and also ascertain its applicability bounds.

In §§ 6.2 and 6.3 we study integro-local theorems for sums of r.v.'s with light tails of the form

$$\mathbf{P}(\xi \geqslant t) = e^{-\lambda_+ t} V(t), \qquad 0 < \lambda_+ < \infty,$$

where $V(t)$ is a function regularly varying as $t \to \infty$. In a number of cases the methods presented in § 6.1 do not work for such distributions, but one can achieve success using the results of Chapters 3 and 4. In § 6.2 we consider the case when the index of the function $V(t)$ belongs to the interval $(-1, -3)$ (in this case one uses the results of Chapter 3); in § 6.3, we take the index of $V(t)$ to be less than -3 (in this case one needs the results of Chapter 4).

In § 6.4 we consider large deviations in more general boundary problems. However, here the exposition has to be restricted to several special types of boundary $\{g(k)\}$, as the nature of the boundary-crossing probabilities turns out to be quite complicated and sensitive to the particular form of the boundary.

Chapters 7–16 are devoted to some more specialized aspects of the theory of random walks and also to some generalizations of the results of Chapters 2–5 and their extensions to continuous-time processes.

In Chapter 7 we continue the study of functions of subexponential distributions that we began in § 1.4. Now, for narrower classes of regularly varying and semiexponential distributions, we obtain wider conditions enabling one to find the desired asymptotics (§§ 7.1 and 7.2). In § 7.3 we apply the obtained results to study the asymptotics of the distributions of stopped sums and their maxima, i.e. the asymptotics of

$$\mathbf{P}(S_\tau \geqslant x) \qquad \text{and} \qquad \mathbf{P}(\overline{S}_\tau \geqslant x),$$

where the r.v. τ is either independent of $\{S_k\}$ or is a stopping time for that sequence (§§ 7.3 and 7.4). In § 7.5 we discuss an alternative approach (to that presented in Chapters 3–5) to studying the asymptotics of $\mathbf{P}(\overline{S}_\infty \geqslant x)$ in the case of subexponential distributions of the summands ξ_k with $\mathbf{E}\xi_k < 0$. The approach is based on factorization identities and the results of § 1.3. Here we also obtain integro-local theorems and asymptotic expansions in the integral theorem under minimal conditions (in Chapters 3–5 the conditions were excessive, as there we also included the case $n < \infty$).

Chapter 8 is devoted to a systematic study of the asymptotics of the first hitting time distribution, i.e. the probabilities $\mathbf{P}\big(\eta_+(x) \geqslant n\big)$ as $n \to \infty$, where $\eta_+(x) := \min\{k \geqslant 1 : S_k \geqslant x\}$, and also of similar problems for $\eta_-(x) := \min\{k \geqslant 1 : S_k \leqslant -x\}$. We classify the results according to the following three main criteria:

(1) the value of x, distinguishing between the three cases $x = 0$, $x > 0$ is fixed and $x \to \infty$;
(2) the drift direction (the value of the expectation $\mathbf{E}\xi$, if it exists);
(3) the properties of the distribution of ξ.

In § 8.1 we consider the case of a fixed level x (usually $x = 0$) and different combinations of criteria (2) and (3). In § 8.2 we study the case when $x \to \infty$ together with n, again with different combinations of criteria (2) and (3).

In Chapter 9 the results of Chapters 3 and 4 are extended to the multivariate case. Our attention is given mainly to integro-local theorems, i.e. to studying the asymptotics

$$\mathbf{P}\big(S_n \in \Delta[x)\big),$$

where $S_n = \sum_{j=1}^{n} \xi_j$ is the sum of d-dimensional i.i.d. random vectors and

$$\Delta[x) := \{y \in \mathbb{R}^d : x_i \leqslant y_i < x_i + \Delta\}$$

is a cube with edge length Δ and a vertex at the point $x = (x_1, \ldots, x_d)$. The reason is that in the multivariate case, the language and approach of integro-local theorems prove to be the most natural. Integral theorems are more difficult to prove directly and can easily be derived from the corresponding integro-local theorems.

Another difficulty arising when one is studying the probabilities of large deviations of the sums S_n of 'heavy-tailed' random vectors ξ_k consists in defining and classifying the very concept of a heavy tailed multivariate distribution. In § 9.1, examples are given in which the main contribution to the probability for S_n to hit a remote cube $\Delta[x)$ comes not from trajectories with one large jump (as in the univariate case) but from those with exactly k large jumps, where k can be any integer between 1 and $d > 1$. In § 9.2 we concentrate on the 'most regular' jump distributions and establish integro-local theorems for them, both when $\mathbf{E}|\xi|^2 = \infty$ and when $\mathbf{E}|\xi|^2 < \infty$; § 9.3 is devoted to integral theorems which can be obtained using integro-local theorems as well as in a direct way. In the latter case, one has to impose conditions on the asymptotics of the probability that the remote set under consideration will be reached by one large jump.

We then return to univariate random walks. In Chapter 10 such walks are considered as processes, and we study there the probability of large deviations of such processes in their trajectory spaces. In other words, we study the asymptotics of

$$\mathbf{P}\big(S_n(\cdot) \in xA\big),$$

where $S_n(t) = S_{\lfloor nt \rfloor}$, $t \in [0,1]$, and A is a measurable set in the space $D(0,1)$ of functions without discontinuities of the second type (it is supposed that the set A is bounded away from zero). Under certain conditions on the structure of the set A the desired asymptotics are found for regularly varying jump distributions, both in the case of 'one-sided' sets (§ 10.2) and in the general case (§ 10.3). Here we use the results of Chapter 3 when $\mathbf{E}\xi^2 = \infty$ and the results of Chapter 4 when $\mathbf{E}\xi^2 < \infty$.

Chapters 11–14 are devoted to extending the results of Chapters 3 and 4 to random walks of a more general nature, when the jumps ξ_i are independent but not identically distributed. In Chapter 11 we consider the simplest problem of this kind, the large deviation probabilities of sums of r.v.'s of two different types. In § 11.1 we discuss a motivation for the problem and give examples. As before, we let $S_n := \sum_{i=1}^{n} \xi_i$ and, moreover, $T_m := \sum_{i=1}^{m} \tau_i$, where the r.v.'s τ_i are independent of each other and also of $\{\xi_k\}$ and are identically distributed. We are interested in the asymptotics of the probabilities

$$P(m,n,x) := \mathbf{P}(T_m + S_n \geqslant x)$$

as $x \to \infty$. In § 11.2 we study the asymptotics of $P(m,n,x)$ for the case of regularly varying distributions and in § 11.3 for the case of semiexponential distributions.

In Chapters 12 and 13 we consider random walks with arbitrary non-identically distributed jumps ξ_j in the triangular array scheme, both in the case of an infinite second moment (Chapter 12 contains extensions of the results of Chapter 3) and in the case of a finite second moment (Chapter 13 is a generalization of Chapter 4). The order of exposition in Chapters 12 and 13 is roughly the same as in Chapters 3 and 4. In §§ 12.1 and 13.1 we obtain upper and lower bounds for $\mathbf{P}(\overline{S}_n \geqslant x)$ and $\mathbf{P}(S_n \geqslant x)$ respectively. The asymptotics of the probability of the crossing of an arbitrary remote boundary are found in §§ 12.2, 12.3 and 13.2. Here we also obtain bounds, uniform in a, for the probabilities $\mathbf{P}\big(\overline{S}_n(a) \geqslant x\big)$ and the distributions of the first crossing time of the level $x \to \infty$.

In § 12.4 we establish theorems on the convergence of random walks to random processes. On the basis of these results, in § 12.5 we study transient phenomena in the problem on the asymptotics of the distribution of $\overline{S}_n(a)$ as $n \to \infty$, $a \to 0$. Similar results for random walks with jumps ξ_i having a finite second moments are established in § 13.3.

The results of Chapters 12 and 13 enable us in Chapter 14 to extend the main assertions of these chapters to the case of *dependent* jumps. In § 14.1 we give a description of the classes of random walks that admit an asymptotic analysis in the spirit of Chapters 12 and 13. These classes include:

(1) martingales with a common majorant of the jump distributions;
(2) martingales defined on denumerable Markov chains;
(3) martingales defined on arbitrary Markov chains;

(4) arbitrary random walks defined on arbitrary Markov chains.

For arbitrary Markov chains one can obtain essentially the same results as for denumerable ones, but the exposition becomes much more technical. For this reason, and also because in case (1) one can obtain (and in a rather simple way) only bounds for the distributions of interest, we will restrict ourselves in Chapter 14 to considering martingales and arbitrary random walks defined on denumerable Markov chains.

In § 14.2 we obtain upper and lower bounds for and also the asymptotics of the probabilities $\mathbf{P}(S_n \geqslant x)$ and $\mathbf{P}(\overline{S}_n \geqslant x)$ for such walks in the case where the jumps in the walk can have infinite variance. The case of finite variance is considered in § 14.3. In § 14.4 we study arbitrary random walks defined on denumerable Markov chains.

Chapters 15 and 16 are devoted to extending the results of Chapters 2–5 to continuous-time processes. Chapter 15 contains such extensions to processes $\{S(t)\}$ with independent increments. Two approaches are considered. The first is presented in § 15.2. It is based on using the closeness of the trajectories of the processes with independent increments to random polygons with vertices at the points $(k\Delta, S(k\Delta))$ for a small fixed Δ, where the $S(k\Delta)$ are clearly the sums of i.i.d. r.v.'s that we studied in Chapters 2–5. The second approach is presented in § 15.3. It consists of applying the same philosophy, based on singling out one large jump (now in the process $\{S(t)\}$), as that employed in Chapters 2–5. Using this approach, we can extend to the processes $\{S(t)\}$ all the results of Chapters 3 and 4, including those for asymptotic expansions. The first approach (that in § 15.1) only allows one to extend the first-order asymptotics results.

Chapter 16 is devoted to the generalized renewal processes

$$S(t) := S_{\nu(t)} + qt, \qquad t \geqslant 0,$$

where q is a linear drift coefficient,

$$\nu(t) := \sum_{k=1}^{\infty} \mathbf{1}(\mathbf{t}_k \leqslant t) = \min\{k \geqslant 1 : \mathbf{t}_k \geqslant t\} - 1,$$

$\mathbf{t}_k := \tau_1 + \cdots + \tau_k$, the r.v.'s τ_j are independent of each other and of $\{\xi_k\}$ and are identically distributed with finite mean $a_\tau := \mathbf{E}\tau_1$. It is assumed that the distribution tails $\mathbf{P}(\xi \geqslant t) = V(t)$ of the r.v.'s ξ (and in some cases also the distribution tails $\mathbf{P}(\tau \geqslant t) = V_\tau(t)$ of the r.v.'s τ) are regularly varying functions or are dominated by such. In § 16.2 we study the probabilities of large deviations of the r.v.'s $S(T)$ and $\overline{S}(T) := \max_{t \leqslant T} S(t)$ under the assumption that the mean trend in the process is equal to zero, $\mathbf{E}\xi + q\mathbf{E}\tau = 0$. Here substantial contributions to the probabilities $\mathbf{P}(S(T) \geqslant x)$ and $\mathbf{P}(\overline{S}(T) \geqslant x)$ can come not only from large jumps ξ_j but also from large renewal intervals τ_j (especially when $q > 0$). Accordingly, in some deviation zones, to the natural (and expected) quantity $H(T)V(x)$ (where $H(T) := \mathbf{E}\nu(T)$) giving the asymptotics of $\mathbf{P}(S(T) \geqslant x)$ and $\mathbf{P}(\overline{S}(T) \geqslant x)$, we may need to add, say, values of the form $a_\tau^{-1}(\tau - x/q)V_\tau(x/q)$, which can dominate when $V_\tau(t) \gg V(t)$. The asymptotics of the probabilities $\mathbf{P}(S(T) \geqslant x)$ and $\mathbf{P}(\overline{S}(T) \geqslant x)$ are studied in § 16.2 in a rather exhaustive way: for values of q having both signs, for different relations between $V(t)$ and $V_\tau(t)$ or between x and T and for all the large deviation zones.

In § 16.3 we obtain asymptotic expansions for $\mathbf{P}(S(T) \geqslant x)$ under additional assumptions on the smoothness of the tails $V(t)$ and $V_\tau(t)$. The asymptotics of the probability $\mathbf{P}\big(\sup_{t\leqslant T}(S(t)-g(t)) \geqslant 0\big)$ of the crossing of an arbitrary remote boundary $g(t)$ by the process $\{S(t)\}$ are studied in § 16.4. The case of a linear boundary $g(t)$ is considered in greater detail in § 16.5.

Let us briefly list the main special features of the present book.

1. **The traditional range of problems on limit theorems for the sums S_n is considerably extended in the book**: we include the so-called boundary problems relating to the crossing of given boundaries by the trajectory of the random walk. In particular, this applies to problems, of widespread application, on the probabilities of large deviations of the maxima $\overline{S}_n = \max_{k\leqslant n} S_k$ of sums of random variables.
2. The book is **the first monograph in which the study of the above-mentioned wide range of problems is carried out in a comprehensive and systematic way** and, as a rule, under minimal adequate conditions. It should fill a number of previously existing gaps.
3. In the book, **for the first time in a monograph, asymptotic expansions** (the asymptotics of second and higher orders) **under rather general conditions**, close to minimal, are studied for the above-mentioned range of problems. (Asymptotics expansions for $\mathbf{P}(S_n \geqslant x)$ were also studied in [276] but for a narrow class of distributions.)
4. Along with classical random walks, **a comprehensive asymptotic analysis is carried out for generalized renewal processes.**
5. For the first time in a monograph, **multivariate large deviation problems for jump distributions regularly varying at infinity** are touched upon.
6. For the first time **complete results on large deviations for random walks with non-identically distributed jumps in the triangular array scheme are obtained. Transient phenomena are studied for such walks with jumps having an infinite variance.**

One may also note that the following are included:

- integro-local theorems for the sums S_n;
- a study of the structure of the classes of semiexponential and subexponential distributions;
- analogues of the law of the iterated logarithm for random walks with infinite jump variance;

- a derivation of the asymptotics of $\mathbf{P}(S_n \geqslant x)$ and $\mathbf{P}(\overline{S}_n \geqslant x)$ for random walks with dependent jumps, defined on Markov chains.

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For the reader's attention We use := for 'is defined by', 'iff' for 'if and only if', and □ for the end of proof. Parts of the expositions that are, from our viewpoint, of secondary interest, are typeset in a small font.

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