Advanced Data Structures

*Advanced Data Structures* presents a comprehensive look at the ideas, analysis, and implementation details of data structures as a specialized topic in applied algorithms. This book examines efficient ways to realize query and update operations on sets of numbers, intervals, or strings by various data structures, including search trees, structures for sets of intervals or piecewise constant functions, orthogonal range search structures, heaps, union-find structures, dynamization and persistence of structures, structures for strings, and hash tables. Instead of relegating data structures to trivial material used to illustrate object-oriented programming methodology, this is the first volume to show data structures as a crucial algorithmic topic. Numerous code examples in C and more than 500 references make *Advanced Data Structures* an indispensable text.

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Advanced Data Structures

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Dedicated to my parents,
Gisela and Helmut Brass
Contents

Preface xi

1 Elementary Structures 1
   1.1 Stack 1
   1.2 Queue 8
   1.3 Double-Ended Queue 16
   1.4 Dynamical Allocation of Nodes 16
   1.5 Shadow Copies of Array-Based Structures 18

2 Search Trees 23
   2.1 Two Models of Search Trees 23
   2.2 General Properties and Transformations 26
   2.3 Height of a Search Tree 29
   2.4 Basic Find, Insert, and Delete 31
   2.5 Returning from Leaf to Root 35
   2.6 Dealing with Nonunique Keys 37
   2.7 Queries for the Keys in an Interval 38
   2.8 Building Optimal Search Trees 40
   2.9 Converting Trees into Lists 47
   2.10 Removing a Tree 48

3 Balanced Search Trees 50
   3.1 Height-Balanced Trees 50
   3.2 Weight-Balanced Trees 61
   3.3 (a, b)- and B-Trees 72
   3.4 Red-Black Trees and Trees of Almost Optimal Height 89
   3.5 Top-Down Rebalancing for Red-Black Trees 101
   3.6 Trees with Constant Update Time at a Known Location 111
   3.7 Finger Trees and Level Linking 114
## Contents

3.8 Trees with Partial Rebuilding: Amortized Analysis 119  
3.9 Splay Trees: Adaptive Data Structures 122  
3.10 Skip Lists: Randomized Data Structures 135  
3.11 Joining and Splitting Balanced Search Trees 143  

4 Tree Structures for Sets of Intervals 148  
4.1 Interval Trees 148  
4.2 Segment Trees 154  
4.3 Trees for the Union of Intervals 162  
4.4 Trees for Sums of Weighted Intervals 169  
4.5 Trees for Interval-Restricted Maximum Sum Queries 174  
4.6 Orthogonal Range Trees 182  
4.7 Higher-Dimensional Segment Trees 196  
4.8 Other Systems of Building Blocks 199  
4.9 Range-Counting and the Semigroup Model 202  
4.10 kd-Trees and Related Structures 204  

5 Heaps 209  
5.1 Balanced Search Trees as Heaps 210  
5.2 Array-Based Heaps 214  
5.3 Heap-Ordered Trees and Half-Ordered Trees 221  
5.4 Leftist Heaps 227  
5.5 Skew Heaps 235  
5.6 Binomial Heaps 239  
5.7 Changing Keys in Heaps 248  
5.8 Fibonacci Heaps 250  
5.9 Heaps of Optimal Complexity 262  
5.10 Double-Ended Heap Structures and Multidimensional Heaps 267  
5.11 Heap-Related Structures with Constant-Time Updates 271  

6 Union-Find and Related Structures 278  
6.1 Union-Find: Merging Classes of a Partition 279  
6.2 Union-Find with Copies and Dynamic Segment Trees 293  
6.3 List Splitting 303  
6.4 Problems on Root-Directed Trees 306  
6.5 Maintaining a Linear Order 317  

7 Data Structure Transformations 321  
7.1 Making Structures Dynamic 321  
7.2 Making Structures Persistent 330  

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# Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>Data Structures for Strings</td>
<td>335</td>
</tr>
<tr>
<td>8.1</td>
<td>Tries and Compressed Tries</td>
<td>336</td>
</tr>
<tr>
<td>8.2</td>
<td>Dictionaries Allowing Errors in Queries</td>
<td>356</td>
</tr>
<tr>
<td>8.3</td>
<td>Suffix Trees</td>
<td>360</td>
</tr>
<tr>
<td>8.4</td>
<td>Suffix Arrays</td>
<td>367</td>
</tr>
<tr>
<td>9</td>
<td>Hash Tables</td>
<td>374</td>
</tr>
<tr>
<td>9.1</td>
<td>Basic Hash Tables and Collision Resolution</td>
<td>374</td>
</tr>
<tr>
<td>9.2</td>
<td>Universal Families of Hash Functions</td>
<td>380</td>
</tr>
<tr>
<td>9.3</td>
<td>Perfect Hash Functions</td>
<td>391</td>
</tr>
<tr>
<td>9.4</td>
<td>Hash Trees</td>
<td>397</td>
</tr>
<tr>
<td>9.5</td>
<td>Extendible Hashing</td>
<td>398</td>
</tr>
<tr>
<td>9.6</td>
<td>Membership Testers and Bloom Filters</td>
<td>402</td>
</tr>
<tr>
<td>10</td>
<td>Appendix</td>
<td>406</td>
</tr>
<tr>
<td>10.1</td>
<td>The Pointer Machine and Alternative Computation Models</td>
<td>406</td>
</tr>
<tr>
<td>10.2</td>
<td>External Memory Models and Cache-Oblivious Algorithms</td>
<td>408</td>
</tr>
<tr>
<td>10.3</td>
<td>Naming of Data Structures</td>
<td>409</td>
</tr>
<tr>
<td>10.4</td>
<td>Solving Linear Recurrences</td>
<td>410</td>
</tr>
<tr>
<td>10.5</td>
<td>Very Slowly Growing Functions</td>
<td>412</td>
</tr>
<tr>
<td>11</td>
<td>References</td>
<td>415</td>
</tr>
<tr>
<td></td>
<td>Author Index</td>
<td>441</td>
</tr>
<tr>
<td></td>
<td>Subject Index</td>
<td>455</td>
</tr>
</tbody>
</table>
This book is a graduate-level textbook on data structures. A data structure is a method1 to realize a set of operations on some data. The classical example is to keep track of a set of items, the items identified by key values, so that we can insert and delete (key, item) pairs into the set and find the item with a given key value. A structure supporting these operations is called a dictionary. Dictionaries can be realized in many different ways, with different complexity bounds and various additional operations supported, and indeed many kinds of dictionaries have been proposed and analyzed in literature, and some will be studied in this book.

In general, a data structure is a kind of higher-level instruction in a virtual machine: when an algorithm needs to execute some operations many times, it is reasonable to identify what exactly the needed operations are and how they can be realized in the most efficient way. This is the basic question of data structures: given a set of operations whose intended behavior is known, how should we realize that behavior?

There is no lack of books carrying the words “data structures” in the title, but they merely scratch the surface of the topic, providing only the trivial structures stack and queue, and then some balanced search tree with a large amount of handwaving. Data structures started receiving serious interest in the 1970s, and, in the first half of the 1980s, almost every issue of the Communications of the ACM contained a data structure paper. They were considered a central topic, received their own classification in the Computing Subject Classification,2

1 This is not a book on object-oriented programming. I use the words “method” and “object” in their normal sense.

2 Classification code: E.1 data structures. Unfortunately, the Computing Subject Classification is too rough to be useful.
and became a standard part of computer science curricula. Wirth titled a book *Data Structures + Algorithms = Programs*, and *Algorithms and Data Structures* became a generic textbook title. But the only monograph on an algorithmic aspect of data structures is the book by Overmars (1983) (which is still in print, a kind of record for an LNCS series book). Data structures received attention in a number of application areas, foremost as index structures in databases. In this context, structures for geometric data have been studied in the monographs of Samet (1990, 2006); the same structures were studied in the computer graphics context in Langetepe and Zachmann (2006). Recently, motivated by bioinformatics applications, string data structures have been much studied. There is a steady stream of publications on data structure theory as part of computational geometry or combinatorial optimization. But in the numerous textbooks, data structures are only viewed as an example application of object-oriented programming, excluding the algorithmic questions of how to really do something nontrivial, with bounds on the worst-case complexity. It is the aim of this book to bring the focus back to data structures as a fundamental subtopic of algorithms. The recently published *Handbook of Data Structures* (Mehta and Sahni 2005) is a step in the same direction.

This book contains real code for many of the data structures we discuss and enough information to implement most of the data structures where we do not provide an implementation. Many textbooks avoid the details, which is one reason that the structures are not used in the places where they should be used. The selection of structures treated in this book is therefore restricted almost everywhere to such structures that work in the pointer-machine model, with the exception of hash tables, which are included for their practical importance. The code is intended as illustration, not as ready-to-use plug-in code; there is certainly no guarantee of correctness. Most of it is available with a minimal testing environment on my homepage.

This book started out as notes for a course I gave in the 2000 winter semester at the Free University Berlin; I thank Christian Knauer, who was my assistant for that course: we both learned a lot. I offered this course again in the fall semesters of 2004–7 as a graduate course at the City College of New York and used it as a base for a summer school on data structures at the Korean Advanced Institute of Science and Technology in July 2006. I finished this book in November 2007.

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3 ABET still lists them as one of five core topics: algorithms, data structures, software design, programming languages, and computer architecture.
Preface

I thank Emily Voytek and Günter Rote for finding errors in my code examples, Otfried Cheong for organizing the summer school at KAIST, and the summer school’s participants for finding further errors. I thank Christian Knauer and Helmut Brass for literature from excellent mathematical libraries at the Free University Berlin and Technical University Braunschweig, and János Pach for access to the online journals subscribed by the Courant Institute. A project like this book would not have been possible without access to good libraries, and I tried to cite only those papers that I have seen.

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Basic Concepts

A data structure models some abstract object. It implements a number of operations on this object, which usually can be classified into

– creation and deletion operations,
– update operations, and
– query operations.

In the case of the dictionary, we want to create or delete the set itself, update the set by inserting or deleting elements, and query for the existence of an element in the set.

Once it has been created, the object is changed by the update operations. The query operations do not change the abstract object, although they might change the representation of the object in the data structure: this is called an adaptive data structure – it adapts to the query to answer future similar queries faster.

Data structures that allow updates and queries are called dynamic data structures. There are also simpler structures that are created just once for some given object and allow queries but no updates; these are called static data structures. Dynamic data structures are preferable because they are more general, but we also need to discuss static structures because they are useful as building blocks for dynamic structures, and, for some of the more complex objects we encounter, no dynamic structure is known.

We want to find data structures that realize a given abstract object and are fast. The size of structures is another quality measure, but it is usually of less importance. To express speed, we need a measure of comparison; this is the size of the underlying object, not our representation of that object. Notice that a long sequence of update operations can still result in a small object. Our
usual complexity measure is the worst-case complexity; so an operation in a specific data structure has a complexity $O(f(n))$ if, for any state of the data structure reached by a sequence of update operations that produced an object of size $n$, this operation takes at most time $Cf(n)$ for some $C$. An alternative but weaker measure is the amortized complexity; an update operation has amortized complexity $O(f(n))$ if there is some function $g(n)$ such that any sequence of $m$ of these operations, during which the size of the underlying object is never larger than $n$, takes at most time $g(n) + mCf(n)$, so in the average over a long sequence of operations the complexity is bounded by $Cf(n)$.

Some structures are randomized, so the data structure makes some random choices, and the same object and sequence of operations do not always lead to the same steps of the data structure. In that case we analyze the expected complexity of an operation. This expectation is over the random choices of the data structure; the complexity is still the worst case of that expectation over all objects of that size and possible operations.

In some situations, we cannot expect a nontrivial complexity bound of type $O(f(n))$ because the operation might give a large answer. The size of the answer is the output complexity of the operation, and, for operations that sometimes have a large output complexity, we are interested in output-sensitive methods, which are fast when the output is small. An operation has output-sensitive complexity $O(f(n) + k)$ if, on an object of size $n$ that requires an output of size $k$, the operation takes at most time $C(f(n) + k)$.

For dynamic data structures, the time to create the structure for an empty object is usually constant, so we are mainly interested in the update and query times. The time to delete a structure of size $n$ is almost always $O(n)$. For static data structures we already create an object of size $n$, so there we are interested in the creation time, known as preprocessing time, and the query time.

In this book, $\log_a n$ denotes the logarithm to base $a$; if no base is specified, we use base 2.

We use the Bourbaki-style notation for closed, half-open, and open intervals, where $[a, b]$ is the closed interval from $a$ to $b$, $]a, b[$ is the open interval, and the half-open intervals are $[a, b[$, missing the first point, and $]a, b]$, missing the last point.

Similar to the $O(\cdot)$-notation for upper bounds mentioned earlier, we also use the $\Omega(\cdot)$ for lower bounds and $\Theta(\cdot)$ for simultaneous upper and lower bounds. A nonnegative function $f$ is $O(g(n))$, or $\Omega(g(n))$, if for some positive $C$ and all sufficiently large $n$ holds $f(n) \leq Cg(n)$, or $f(n) \geq Cg(n)$, respectively. And $f$ is $\Theta(g(n))$ if it is simultaneously $O(g(n))$ and $\Omega(g(n))$. Here “sufficiently large” means that $g(n)$ needs to be defined and positive.
The code examples in this book are given in standard C. For the readers used to some other imperative programming language, most constructs are self-explanatory.

In the code examples, = denotes the assignment and == the equality test. Outside the code examples, we will continue to use = in the normal way.

The Boolean operators for “not,” “and,” “or” are !, &&, ||, respectively, and % denotes the modulo operator.

Pointers are dereferenced with *, so if pt is a pointer to a memory location (usually a variable), then *pt is that memory location. Pointers have a type to determine how the content of that memory location should be interpreted. To declare a pointer, one declares the type of the memory location it points to, so “int *pt;” declares pt to be a pointer to an int. Pointers are themselves variables; they can be assigned, and it is also possible to add integers to a pointer (pointer arithmetic). If pt points to a memory object of a certain type, then pt+1 points to the next memory location for an object of that type; this is equivalent to treating the memory as a big array of objects of that type. NULL is a pointer that does not point to any valid memory object, so it can be used as a special mark in comparisons.

Structures are user-defined data types that have several components. The components themselves have a type and a name, and they can be of any type, including other structures. The structure cannot have itself as a type of a component, because that would generate an unbounded recursion. But it can have a pointer to an object of its own type as component; indeed, such structures are the main tool of data structure theory. A variable whose type is a structure can be assigned and used like any other variable. If z is a variable of type C, and we define this type by

    typedef struct { float x; float y; } C,

then the components of z are z.x and z.y, which are two variables of type float. If zpt is declared as pointer to an object of type C (by C *zpt;), then the components of the object that zpt points to are (*zpt).x and (*zpt).y. Because this is a frequently used combination, dereferencing a pointer and selecting a component, there is an alternative notation zpt->x and zpt->y. This is equivalent, but preferable, because it avoids the operator priority problem: dereferencing has lower priority than component selection, so (*zpt).x is not the same as *zpt.x.

We avoid writing the functions recursively, although in some cases this might simplify the code. But the overhead of a recursive function call is significant
and thus conflicts with the general aim of highest efficiency in data structures. We do not practice any similar restrictions for nonrecursive functions; a good compiler will expand them as inline functions, avoiding the function call, or they could be written as macro functions.

In the text we will also frequently use the name of a pointer for the object to which it points.