Introduction: an overview

1. Rigidity in dynamics

In a very general sense, modern theory of smooth dynamical systems deals with smooth actions of “sufficiently large but not too large” groups or semigroups (usually locally compact but not compact) on a “sufficiently small” phase space (usually compact, or, sometimes, finite volume manifolds). Important branches of dynamics specifically consider actions preserving a geometric structure with an infinite-dimensional group of automorphisms, two principal examples being a volume and a symplectic structure. The natural equivalence relation for actions is differentiable (corr. volume preserving or symplectic) conjugacy.

One version of the general notion of rigidity in this context would refer to a certain class $\mathcal{A}$ of actions being described by a finite set of parameters, usually smooth *moduli*. Examples of such classes are all actions in the neighborhood of a given one, or all actions of a continuous group with the same orbits, or all G-extensions of a given action α to a given principal G-bundle. In some situations this is too strong and, rather than classifying all actions from $\mathcal{A}$, one may require that actions equivalent to a given one have a finite codimension in a properly defined sense, e.g., appear in typical or generic finite-parametric families of actions.

A perfect extreme case appears when all actions from $\mathcal{A}$ belong to a single equivalence class. This may be referred to as rigidity in the narrow sense of the word. However, one should point out that for abelian group actions even locally this can only happen in the case of discrete groups, since otherwise one can always compose the original action with a group automorphism close to the identity.

2. Limited extent of rigidity in traditional dynamics

The material presented in this book relies to a considerable extent on the classical theory of (uniform) hyperbolic and partially hyperbolic systems, i.e., the study of *rank-one* cases of $\mathbb{Z}_+$, $\mathbb{Z}$, and $\mathbb{R}$-actions with hyperbolic or partially hyperbolic behavior. Anosov diffeomorphisms and Anosov flows (see Section 1.8.1) are prime examples of such actions.

Anosov diffeomorphisms and Anosov flows display certain elements of rigidity of the *topological* orbit structure, both locally (structural stability, see e.g., [67, Corollary 18.2.2]) and globally (topological restrictions on the ambient manifolds and homotopy invariants [67, Theorem 18.6.1]). Nevertheless, their *differentiable* properties are far from rigid; at best, the classification with respect to differentiable conjugacy is given by infinitely many moduli as in the case of Anosov diffeomorphisms of $\mathbb{T}^2$ ([67, Section 20.4]), or transitive Anosov flows on three-dimensional manifolds ([105]). Similarly, cohomology classes of sufficiently regular (Hölder or smooth) rank-one cocycles over an Anosov system are determined by infinitely many parameters, e.g., the periodic data (see Theorem 4.2.2).

It is also worth pointing out that local differentiable rigidity in the more loose parametric sense does take place for some non-hyperbolic systems, such as circle rotations with Diophantine rotation number or, more generally, for Diophantine translations and linear flows on a torus.

3. Rigidity for actions of higher rank abelian groups

The goal of this monograph and its projected sequel is to give an up-to-date and, as much as possible, self-contained presentation of certain rigidity phenomena that appear for actions of higher rank abelian groups by smooth maps on compact differentiable manifolds. We will consider hyperbolic and partially hyperbolic $\mathbb{Z}^k$- and $\mathbb{R}^k$-actions, $k \geq 2$, or, more generally, $\mathbb{Z}^k \times \mathbb{R}^l$, $k+l \geq 2$.

Certain results for higher rank abelian semigroups $\mathbb{Z}_+^k$, $k \geq 2$, are also discussed and, by a slight abuse of terminology, we will use the phrase *higher rank abelian groups actions* for them as well.

The list of known examples of Anosov (normally hyperbolic) and partially hyperbolic actions of higher rank abelian groups which do not arise from products and other standard constructions is restricted. All such basic examples are differentiably conjugate to algebraic actions. Basic definitions appear in Section 1.6; principal classes of examples are surveyed in Chapter 2. These actions exhibit a remarkable array of measurable and differentiable rigidity properties, markedly different from the rank-one situation. In this and the subsequent volume we are concerned with differentiable properties, namely:

(i) *cocycle rigidity;*
(ii) *local differentiable rigidity*, including *foliation rigidity;*
(iii) *global differentiable rigidity.*

In this volume we describe the scene in sufficient detail, and develop principal methods which are at present used in various aspects of the rigidity theory. Part I serves as an exposition and preparation. Cocycle rigidity, which occupies Part II of this volume, serves both as a model for other rigidity phenomena and as a tool for studying them.

The area of differentiable rigidity is experiencing a rapid development. While local differentiable rigidity for Anosov algebraic actions was proved in the 1990s [81], now we are close to a comprehensive understanding of local differentiable rigidity for all appropriate classes of partially hyperbolic algebraic actions; see [19, 22, 23, 171, 172] for a partial realization of the program. However, a number of key results in the area have not yet appeared in the journals so it is not currently possible to provide a comprehensive treatment in book form. Such a treatment of local differentiable rigidity will appear in the sequel to this book, which will be largely based on the material of the present volume.

We should add that some of the methods developed in the local rigidity theory for partially hyperbolic actions turned out to be applicable to some classes of non-hyperbolic, specifically parabolic actions, thus leading to a totally new phenomena [24].

Global differentiable rigidity has been shown for a certain class of Anosov actions on the torus [151] and for actions satisfying certain dynamical assumptions stronger than just Anosov [64]. A promising opening in the direction of global rigidity is provided by the the non-uniform measure rigidity discussed in the Section 5 below.

4. Mechanisms of rigidity

The following property appears in all known cases and may even turn out to be necessary and sufficient for a (properly adjusted) cocycle rigidity and local differentiable rigidity for an algebraic action of $\mathbb{Z}^k \times \mathbb{R}^l$:

($\mathfrak{R}$) The group $\mathbb{Z}^k \times \mathbb{R}^l$ contains a subgroup L isomorphic to $\mathbb{Z}^2$ such that for the suspension of the restriction of the action to L every element other than the identity

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Anatole Katok , Viorel Nițică
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acts ergodically with respect to the standard invariant measure obtained from the Haar measure.

The basic source of various kinds of rigidity in actions of higher rank abelian groups with hyperbolic behavior is the interplay between the linear algebra, describing the infinitesimal speeds of growth in various directions in time and space, and the existence of recurrence. The key notion here is that of the *Weyl chamber*, which is a generalization of the classical notion from Lie group theory. The reason for the appearance of rigidity in higher rank (encapsulated in the condition ($\mathfrak{R}$)) is that while the dynamics along the walls of Weyl chambers may be (and often is) highly non-trivial, it acts as isometry on a certain invariant foliation (such as a *Lyapunov* or *coarse Lyapunov* foliation, see Section 1.6.4), and hence ties invariant geometric structures on different leaves of that foliation.

In contrast, in the rank-one case, there are only two Weyl chambers, the positive and negative half-lines, and their common boundary is zero. Hence nothing happens along the "wall."

There are certain differences in the treatment of the continuous, discrete invertible, and discrete non-invertible actions. There are some advantages in looking into discrete time situations, primarily better visualization in low dimensions, and we will take this approach while treating some important examples or model problems. However, there are two decisive reasons for taking $\mathbb{R}^k$ as the main case: (i) the geometry of the Weyl chambers, whose walls are often "irrational" and hence "invisible" in the group itself in the discrete time cases, and (ii) the possibility to reduce the other cases to this one via constructions of *natural extension* and *suspension*, which are described in Section 1.2.

5. Measure rigidity

A related class of rigidity phenomena is the rigidity of invariant measures, sometimes called simply *measure rigidity*. There are two directions here: one dealing with algebraic actions, and the other with actions defined by some global topological conditions or with invariant measures satisfying certain dynamical properties.

a. Algebraic actions

One considers here the same class of algebraic Anosov actions as well as their non-Archimedean counterparts. The goal is to classify all Borel invariant measures for such actions and to show specifically that those measures are all of algebraic nature except for very special exceptional cases. This program, which is has been partially realized in [28, 29, 30, 31, 80, 154], can be considered as a counterpart of cocycle rigidity and local differentiable rigidity.

This work uses the same fundamental structures outlined in the previous section which are responsible for cocycle and local differentiable rigidity. There are, however, important technical differences.

For example, harmonic analysis methods are very fruitful in our setting but so far have been much less productive in the study of invariant measures, since it is very difficult to distinguish invariant measures among usually much more abundant invariant distributions.

Another difference appears in the use of invariant structures on various invariant foliations for the action. In the case of a measure such structures are corresponding conditional measures, which may be trivial if the measure in question has zero entropy for all elements of the action. This makes all existing results in measure rigidity for hyperbolic or partially hyperbolic actions subject to an assumption of positivity of entropy and thus fundamentally incomplete.[1] This is reflected in applications. For example, while full measure rigidity for the Weyl chamber flow on $SL(3,\mathbb{R})/SL(3,\mathbb{Z})$ would imply the Littlewood conjecture in multiplicative Diophantine approximation, the existing results for the rigidity of positive entropy measure only imply that the hypothetical set of counter-examples has a Hausdorff dimension of zero [30].

b. Non-uniform measure rigidity

This is a new direction based on combining geometric ideas of measure rigidity with those of non-uniform hyperbolicity that are mentioned in Section 1.7 [61, 62, 74, 75]. It is still in the process of rapid development and its potential is far from having been realized. However, even the results obtained so far are fairly striking: purely topological conditions on an action lead to the existence of an absolutely continuous invariant measure and a flat affine structure defined on an invariant set of positive volume. Furthermore, there is a smooth conjugacy in the sense of Whitney correspondence on an invariant set of positive volume with a standard algebraic model. This opens a new approach to global differentiable rigidity problems: for globally hyperbolic actions, invariant geometric structures smooth in the sense of Whitney probably can be extended to genuine smooth structures defined everywhere.

[1] This of course stands in contrast with Ratner classification of invariant measures for parabolic unipotent homogeneous actions [84, 148, 178]. It seems that the higher rank hyperbolic case is fundamentally more difficult in this respect.

6. Contrast and similarities with actions of "large" groups

Some of the rigidity phenomena exhibited by smooth actions of higher rank abelian groups with hyperbolic or partially hyperbolic behavior look quite similar to those found in actions of "large" and "rigid" non-abelian groups, such as semisimple Lie groups of $\mathbb{R}$-rank greater than one or irreducible lattices in semisimple Lie groups of $\mathbb{R}$-rank greater than one. Those properties are the main subject of the survey [32], see especially Section 6 there.

For classes of non-abelian groups mentioned above there are fundamental rigidity phenomena already at the measure-theoretic level. The prototype result, fundamental for dynamical applications, is Zimmer's cocycle super-rigidity extension of the Margulis super-rigidity theorem, see [32, Section 6.2], [179]. Based on these fundamental properties, extra geometric, analytical, and dynamical tools allow the study of rigidity properties specific for smooth actions, see [36, 41, 111], for characteristic results in that direction. A recent example of the successful application of the approach based on non-uniform measure rigidity for actions of higher rank abelian groups to rigidity of actions of "large" groups appeared in [76].

Since higher rank abelian groups are amenable, for actions of such groups there are no general rigidity properties at the basic measurable level, such as the classification of measurable cocycles or orbit equivalence [47, Theorem 3.5.4]. Rigidity only appears in the presence of an extra structure, most typically in the smooth case in the presence of certain hyperbolicity. The toolkit has many similarities with that used for going from measurable to differentiable rigidity results for actions of simple Lie groups of $\mathbb{R}$-rank greater than one or irreducible lattices in semisimple Lie groups of $\mathbb{R}$-rank greater than one, see [32, Theorem 6.5.3] as a characteristic example.

7. Background, references, and other sources

We extensively use background material from several areas of mathematics. Let us mention more important ones:

Hyperbolic dynamics occupies the first place here. It is an important field of modern mathematics and expositions of it can be found in numerous places. A standard reference is [67], and a detailed survey of the principal results in the field appears in [46]. Section 1.8 contains formulations of essential results both in the classical case (diffeomorphisms and flows) and for actions of higher rank abelian groups, which are used later.

The theory of Lie groups and Lie algebras extending to symmetric spaces and lattices is essential for understanding classes of algebraic actions which play a central part in our considerations. A comprehensive source for the semisimple case and symmetric spaces is [48]. For the general and nilpotent case one can consult [132]. We choose to present necessary material from this area piecemeal as needed rather than put it in a single place upfront. Necessary material from the general and nilpotent theory is introduced in two places in Section 2.1, while more general results and the semisimple case are reviewed in Section 2.3.3. Lattices in Lie groups, both semisimple and nilpotent, appear throughout our discussion. Fortunately, we do not need much general theory and in most cases co-compact lattices appear as "black boxes." In a number of places they come alive through ingenuous specific constructions.

Classical analysis is essential in the treatment of the regularity of conjugacies and cocycles for the actions considered in this book. A variety of results allow us to conclude the regularity of functions from various seemingly weaker properties as well as to obtain appropriate norm estimates. Some of these results were proved specifically for dynamical applications and were never collected in a single place. This is a crucial part of technical apparatus in rigidity theory and we dedicate the whole of Chapter 4 to a detailed presentation of these results with complete proofs.

Notions and results from other areas appear in a more limited way. Let us mention several of them together with some relevant bibliography: *algebraic number theory* (Section 2.2.4) [16], *algebraic K-theory* (Section 4.4.5) [115], *commutative algebra* (Section 2.4.2) [90], and *theory of unitary representation of Lie groups* (Section 4.4.2) [85].

The area which is the subject of the present book is in the process of active development and expository literature is still quite small. For cocycle rigidity there is a survey by Niţică and Török [128]. Local rigidity is covered in detail in a survey by Fisher [34]. A great overview of measure rigidity by Lindenstrauss [95] contains a few proofs. A more limited and less up-to-date but quite detailed exposition of measure rigidity can be found in the article of Kalinin and Katok [60].

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Part I

Preliminaries from dynamics and analysis

1

Definitions and general properties of abelian group actions

1.1 Group actions, conjugacy, and related notions

Let X be a space provided with a certain structure $\mathcal{S}$; the cases of interest for us are measure (such that X is a Lebesgue space) or an equivalence class of such measures, metrizable topology (usually compact), the structure of a (usually compact) differentiable manifold, or the structure of a homogeneous or double coset space of a Lie group. In each of these cases there exists a natural topology in the space of automorphisms of $\mathcal{S}$. By the action of a topological group G in this context we will always mean a continuous homomorphism α into the space of automorphisms of the structure $\mathcal{S}$. In our setting the group G will always be locally compact; in fact we may assume that it is a Lie group which includes both discrete and connected cases.

An *isomorphism* or *conjugacy* between two actions of a group G, say $\alpha\colon G \times X \to X$ and $\alpha'\colon G \times Y \to Y$, is a bijection $h\colon X \to Y$ that preserves or respects the particular structure (diffeomorphism, homeomorphism, measure-preserving, non-singular map, etc.), such that

$$h(\alpha(g, x)) = \alpha'(g, (h(x)) \text{ for all } g \in G, x \in X. \qquad (1.1.1)$$

The notion of isomorphism is natural from the categorical point of view and provides the natural starting point for looking into the classification of actions. One should note that, in some particular settings, a weaker structure should be preserved in order to have a meaningful working notion. For example, as we mentioned in Section 2 of the introduction, for smooth dynamical systems in the classical setting (i.e., actions of $\mathbb{Z}$ or $\mathbb{R}$), topological classification is in many situations more tractable than a smooth one. Nevertheless, the main purpose of this book is to investigate certain special situations when the classification of actions up to a differentiable conjugacy becomes feasible.