Part 1

Fundamentals

Fundamentals are the psychological entry to a subject, and foundations are the logical entry. Fundamentals must be easy for the novice, while foundations can be hard for the expert. Learning a subject is movement from fundamentals to foundations.

Consistency is desirable, but sometimes psychological or logical inconsistency is a better way to learn. Stochastic theory may begin as a study of probability in finite gambling games, but later expectation is the core, and probability is recast as the expectation of a characteristic function. This is psychologically inconsistent. A logical inconsistency exists in the extension of classical to relativistic mechanics, but it is so psychologically compelling that this point is often ignored. In each case, there is an explanation in the latter theory as to why the former theory seemed to be right at first. This is justification in retrospect.

The material here is designed to be internally logically and psychologically consistent but also to lead naturally from an instinctive human beginning. Although misleading in a number of aspects, there exists later material that not only corrects the errors but also explains the relation of this to higher material and why it looked the way it did but could not really be that way.

The approach used here is not always orthodox, even when covering orthodox material. The intention is to provide psychological and logical hooks on which further development can occur and to avoid psychological pitfalls that could complicate later development.

This part is metalogic to the languages part, which in turn is metalogic to the specification part. Logical language speaks about specification language, which speaks about programming language. But it is full circle, because all of these are software, and so software speaks about software, and the book speaks about itself. There never really was an escape to a metalevel.

This book is an example of what this book is talking about.

1

Arithmetic

It also encourages us to look into our own minds to find out how we reason. This is useful, because our own mind is our first model for software.

We learn arithmetic so young that we accept it as physical reality rather than theory. But it is theory, the first symbolic logic we learn. Recalling how it became so natural is a good step toward understanding software, which is, above all, a formal system. Operations with goats can be performed in proxy on pebbles and in turn by pencil marks. A dot does not need to mean goat all the time as long as the eventual result can be applied. Numerals in computation are symbols without meaning and are no different in kind to generic algebra.

How many goats are in that field? As many as in the other field? It sounds like a simple question, but numbers are not self-evident. If *number* means something physical, the number of goats in that field, then the theory of numbers must be tested by experiment. Counting waves, clouds, or electrons shows that nontrivial assumptions are used when applying the theory of numbers. Assumptions are so familiar that it is difficult to slow down enough to realize they are applied habitually. But it is habit and not instinct. Instinctively, humans understand only a few. Technology is needed to handle more than a handful of grapes. How is a technology of numbers developed when such a technology does not already exist?[1]

These assumptions are known as material conjectures.

Humans do not understand numbers; they understand numerals.[2]

For example, no one understands 237 marbles. If 237 marbles are poured onto the floor from a bag, the human response is that there are a lot of marbles, not 237 marbles. Larger numbers than 5 or 10 are understood in terms of *expressions*. It is expressions that are manipulated to reason about number, not the numbers. Expressions exist in their own right and have their own rules of manipulation. They are separate from numbers, and the user of the technology must understand how results obtained by manipulation of these expressions are related to the physical reality that they are intended to model.

The concept of number here is the physical collection of goats, not the philosophical abstraction.

Arithmetic is just one formal symbolism used with numbers; in addition, algebra is used. Arithmetic describes the behavior of goats in a field, and algebra describes the behavior of numerals on the page. But algebra is also made from manipulation of expressions. Goats, numerals, and formulas are physical, concrete things that humans manipulate to their own ends.

1.1 Natural numbers

Both adjectives *Arabic* and *Hindu* are misleading.

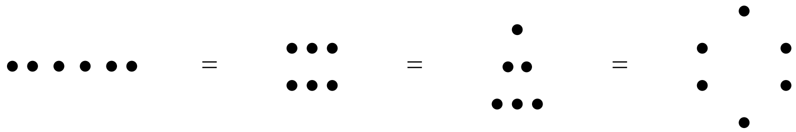
Pencil and paper or abacus, not electronic calculators.

Many modern students might not have to *imagine* this.

A precise definition of *regular* is not intended.

The reader might be familiar with Arabic numerals and Hindu algorithms: 123, 876, 986, and some methods of adding, subtracting, multiplying, and dividing them. This section seeks the origin of this material; seeking is easier if the reader has little familiarity. Readers are asked to imagine they have not yet learned about numerals and arithmetic. How are these concepts to be defined? How can a symbolic system be built from first observational principles to mimic properties of numbers?

Identical small black pebbles •• • • •• can be arranged in a variety of regular patterns. One regular pattern is a straight line, and there are others. Two patterns are numerically equal if the pebbles in one can be rearranged into the other without having any pebbles left over. The symbol = is used to express this property.



Every pattern can be rearranged into a straight line, so numerical equality can be stated as *two patterns can be rearranged into two regular lines (one each) so that the pebbles are paired.*

equal   not equal

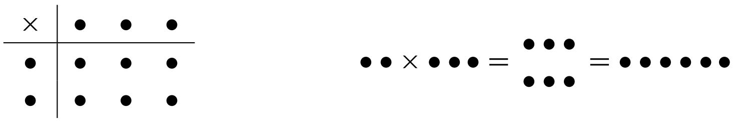
There is no such thing as *absolute* or primary equality.

Numerical equality and geometric equality are distinct. Two patterns of pebbles are geometrically equal if one can be moved onto the other without changing the distance between any two points. If no two points occupy the same space, then any geometrically equal patterns are numerically equal, but the converse does not follow. The concept of equality is neither singular nor simple.

The operation of (material) addition is to put two patterns together.

••• + •••• = •••••••

The operation of (material) multiplication is to fill up a table with pebbles:



Any collection of pebbles can be arranged in a regular line. Which other patterns, rectangles, squares, triangles it can also be arranged in is a material property. Some collections of pebbles can be arranged into a regular square grid, and others cannot. Some can be placed into several regular rectangular grids, and others can be placed only into the degenerate rectangle: the straight line.

These concepts are prelogical. The development above is not the only option, but it is not known how to avoid altogether similar appeals to material intuition in the building of the foundations of arithmetic.

Small black pebbles were chosen to emphasize that the details of the pebbles can be ignored. There is no requirement that the pebbles be identical, as long as the differences are ignored. Red, green, square, star-shaped – it is only important that the pebbles can be placed into patterns.

One way is to use visual imagery, which is an on-board graphics coprocessor in the human brain.

How can these collections be thought about? Is it possible to determine for a large collection, without actually trying it out, whether it can be arranged into a square? Is there a method of writing symbols corresponding to the collections so that manipulation of the symbols will give answers, without manipulating the collections themselves, to questions about the collections.

1.2 Roman numerals

David Hilbert founded mathematics on sequences of vertical strokes. In his metamathematics, only finitary process is accepted, and all is reduced by finitary process to the unavoidable primary intuition of sequence.

Represent • by I, •• by II, • • • by III, and so on. Given a collection represented by IIII and another by III, it is clear that the collection formed by putting all the pebbles together is IIIIIII. Similarly, determine whether the resulting collection can be put into a two-row rectangle by entering the symbols into a table, filling the rows alternately, and we find

```
I I I I
I I I
```

The *subtractive* principle is not being used. It is a bad idea and will not be mentioned further.

The point is not trivial; some cultures mistrusted this argument and insisted on direct purchase of each goat.

This representation is no improvement over the pebbles, other than being *pencil-and-paper*. More work is needed. Introduce another symbol, V; let it represent • • • • •; that is, V=• • • • •. So, • • • • • • • becomes VII, and VII+V is represented by VIIV. That is, VII + V = VIIV. Addition using this symbolic notation is about five times faster than with the pebbles.

You are a merchant with goats to sell at one copper coin each. The number of coins needs to be the same as the number of goats. The direct method is to place one coin each in a bag on the neck of each goat. If the bags are collected and then opened, a new principle is used: if two collections (the goats and coins) are each paired with a third (the bags), then the original two collections must be the same number. It is easier if the goats could be driven through a gate and the buyer hands you a copper coin each time a goat goes through.

Both you and the buyer are from high society and must not be seen handling goats and would prefer to sit in the shade eating dates. Neither you nor the buyer trusts servants with copper coins, but you trust each other. So your servants perform the chosen ritual with pebbles instead. Then the bag of pebbles is brought to you, and you lay out the pebbles and copper coins and pair them. There is the same number of coins as pebbles and pebbles as goats, so there is the same number of coins as goats. But today the buyer, in expectation, has already counted the copper coins into a bag and written the number of them, **VIVI**, on the outside of the bag. Your (educated) servant has counted the goats and written **IIIVIIII**. The symbols used are different; how can equality be determined?

Some cultures used clay balls with inscribed symbols but also containing the right number of pebbles as a check.

The **V**s can be collected in one group and the **I**s in another. Then as many times as possible, five **I**s are taken and one **V** is added. Pebbles marked with the symbols can be used. With this justification, this can now be done with symbols. Changing the order[3] of the symbols does not change the number, **IV**=**VI**. A collection of symbols can be sorted so that all the **V**s are on the left and the **I**s on the right. Then using **IIIII**=**V**, the number of **I**s can be reduced to a minimum. This canonical symbol collection is unique for a given collection, so direct comparison determines whether two collections are numerically equal.

Trusting your money to this means trust in symbolic logic. The idea is not obviously right and took centuries to develop and to become part of common culture.

To speed the process, use **X** to mean **VV** and **L** to mean **XXXXX**.

The rules of symbolic pebble logic:

The point of this exercise is to do the manipulation without translation to base 10, for example. Do not try to work out what the symbols mean.

1. The written symbols form a *numeral* for a pebble collection.
2. Equivalent numerals represent the same number of pebbles.
3. A numeral is equivalent to any rearrangement of its symbols.
4. A part of a numeral is a group of symbols in that numeral.
5. Each part is also a numeral.
6. Replacing a part by an equivalent part makes an equivalent numeral.
7. **IIIII** is equivalent to **V**.
8. **VV** is equivalent to **X**.
9. **XXXXX** is equivalent to **L**.

Reduction: sort the large-value symbols to the left and the small to the right. If there is a **IIIII** in the numeral, replace the left-most one with **V**. Repeat this until there are no more **IIIII**. Now similarly replace any **VV** by **X** and then **XXXXX** by **L**. Finally, no reduction rules apply, and the numeral is atomic. The same atom is generated for any equivalent numeral; it is called the *normal form* of the numeral. The method given is systematic, but random application of any of the rules (including sorting) as they apply will produce the same atom. It is unavoidable. A method like this is *confluent*.

Atomic in a quasiclassical Greek sense.

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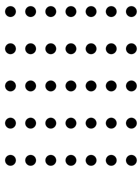
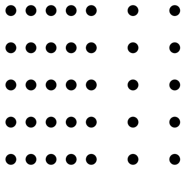
Bookkeeping, not calculation, was the original use of numerals.

In practice, equivalence of numerals can be computed by checking the symbol-by-symbol equality of the atoms produced by these manipulations. Ignoring mistakes, if two people with a pebble collection each count their collections and determine numerals, and then atomic normal forms, then the two collections are numerically equal exactly when the two normal forms are equal. This process is often easier than direct comparison. Another advantage of symbols is that if a mistake is made, the record of the calculation can determine who made the mistake and where.

Usually, a few more symbols are used: **C** for **LL**, **D** for **LLLLL**, and **M** for **DD**. It is possible to reason quickly on paper about collections of goats that would take hours to count.

Now the price of goats goes up. If each goat is now worth • • • • • copper coins and there are • • • • • • • goats, form a rectangular grid and count the pebbles.





How is this done with symbols? • • • • • • • is • • • • •+•+•, so divide the collection in the same manner.

Given **I** collection of **V** columns and **II** collections of **I** column, each **V** columns is **VVVVV** pebbles, which can also be written as **XXV**, and each **I** column is **V** pebbles. So, when **VII** is the number of goats, replace each **V** with **XXV** and each **I** with **V**, obtaining **XXVVV**. This is now normalized to **XXXV**. Similar reasoning shows that if to pay **VVI** each for **VII** goats, take each symbol in the cost numeral and each symbol in the goat count numeral and do the replacement. This is clearer in a table:

	V	V	I
V	XXV	XXV	V
I	V	V	I
I	V	V	I

The contents of the main cell, collected into one numeral, is **XXXXVVVVVVVII**, which is normalized to **LXXVII**. The operation is multiplication, **VVI** × **VVI** = **LXXVII**.

To make this calculation easy, memorize the following table:

	I	V	X	L	C	D	M
I	I	V	X	L	C	D	M
V	V	XXV	L	CCL	D	MMD	MMMMM
X	X	L	C	D	M	MMMMM	
L	L	CCL	D	MMD	MMMMM		
C	C	D	M	MMMMM			
D	D	MMD	MMMMM				
M	M	MMMMM					

The order of combination does not matter, so there is less to memorize than it seems at first glance. The table is also very regular. The lower part is not filled; the numerals are too large. This is, of course, the familiar Roman system of numeration. The Romans invented I' meaning $\mathsf{M} \times \mathsf{X}$, and so on for the higher symbols. This was extended to M'', M''', as required. The entire table above can be filled in with short numerals.

It would have been nice to have a symbol for MMMMM.

The reduction to normal form can be defined by local replacement.

Sorting:

IV→VI					
IX→XI	VX→XV				
IL→LI	LX→LV	XL→LX			
IC→CI	CX→CV	XC→CX	LC→CL		
ID→DI	DX→DV	XD→DX	LD→DL	DC→CD	
IM→MI	MX→MV	XM→MX	LM→ML	MC→CM	DM→MD

Numerical replacement:

IIIII	→	V
VV	→	X
XXXXX	→	L
LL	→	C
CCCCC	→	D
DD	→	M

Each left string is changed into the right string. For example, VVIIX, XIIX, XIXI, XXII.

1.3 Choice of numerals

Roman numeration is mentioned before Arabic because the reader most likely has familiarity with arithmetic in Arabic and a belief that the Roman system is a nightmare. But the nightmare comes from a lack of familiarity. Roman numerals can be added and multiplied efficiently, as above. The ancient Romans most likely never did it this way. But no one else did either at that time. Calculation at the time of the ancient Romans was done by abacus and counting boards, using prototypes of the Hindu logic. The distinction between the Roman numeration and the Hindu numeration is not nearly as strong as it is often made out to be.

Why do we have a place-value system? Was the invention of a place value system a great intellectual achievement of a deep philosopher? Perhaps not. Maybe it was a manufacturing problem. An accountant is working on a counting table. He has a bucket of red, green, and blue balls, valued at 1, 5, and 25. His calculation is like money changing. Change five red balls for one green ball. But he would keep them in separate buckets and separate places on the table. This

"He" is used in the genderless sense.

is a natural response to keep things easy. Now, he runs out of blue balls (on a particularly large sum). He could go to the shop for some more blue balls, or he could put a red ball in the blue ball slot and *call it a blue ball*. After a while, it is realized that it is easier and cheaper to keep a bucket of white balls and know implicitly their "color" from where you put them on the table. This is the place value system.

The logic is almost identical to that in the Arabic base-10 system. The Roman system is effectively a base-10 system insofar as **V** and **I** give the least significant base-10 digit, and **L** and **X** give the next. The Ancient Greeks used a similar system, analogous to using a, b, c, d, e, f, g, h, i, for the digits 1 through 9, and j, k, l, m, n, o, p, q, r, for 10 through 90. The Babylonians, before the Greeks and Romans, used a base-60 system with place value. At first, an empty column was represented by a space and then by a •. But having a place-value base system did not cause the Babylonians to use a Hindu algorithm. The Hindu logic, which can be realized in many systems other than base systems, was developed over several thousands of years in a combined effort by many thinkers.

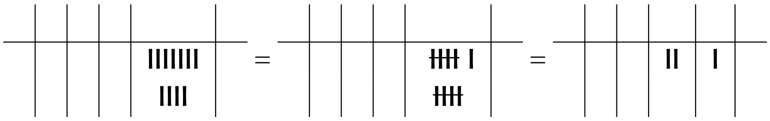
> Very similar to modern Arabic (Hindu) notation.

Variants on • or ○ are almost the only symbols used for zero. Possibly the first is a place holder, a tiny mark to say a digit is missing, and the second represents an empty container. The Hindus did use a symbol like an inverted h, and there is a complex Chinese character for zero, but the rule is broadly correct. For the number 1, either **I** or **_** (possibly bent) is the symbol, and for 2 and 3 the symbols are two or three parallel strokes, written in a rapid, messy, cursive fashion. But from 4 onward, the symbols diverge strongly.

Morris Kline, in *Mathematics in Western Culture*, reports that the Babylonian base-60 system was used in science and mathematics in Europe up until the sixteenth century, so when the base-10 system was introduced, it was not entirely alien. The resistance was not lack of familiarity as such, but rather the existence of viable alternatives and valid concerns about the security of the system itself.

1.4 Tally systems

If there are many **I**s, grouping them, **IIIII IIIII IIIII II**, makes the amount visually clear. A human sees **III** groups of **IIIII** and **II** more. It is fairly natural to change this to **卌**: to draw the fifth stroke horizontally joining them into one typographic unit. Five lots of 5 could be joined into one lot of 25, and so on. A tally table is a regular method of handling this. A mark **I** in a column means **卌** in the column to the right.



This gives a simple method of addition.

	I		IIIIIII IIII

+

		II	IIIII III

=

	I	 II	IIIIIII IIII IIIII III

Which is easier if the strokes are reduced first:

	I	II	I

+

		III	III

=

	I	II III	I III

Which becomes

	II		IIII

Represent the numbers as (;I;II;I) + (;;III;III) = (;;II;;IIII), the semicolons replace the vertical lines.

Because each column counts the 𝍸s in its right-hand neighbor, there is never any *need* to place more than IIII in each column, but there is nothing *wrong* with having more strokes in a column, any more than there is something wrong with having a nonnormalized Roman number.

There is also no need to have each column count the *same* number of strokes in the previous column. The Roman system is essentially a tally system in which the columns alternate between meaning 5 and meaning 2 times the previous column.

1.5 Hindu algorithms

The digits are not given any meaning here other than as abbreviation for the pattern of strokes.

A regular 5-tally layout is normalized when there are at most 4 tally marks in each column. Addition combines the tally marks directly and then normalizes. Normalization starts at the right-most column; each IIIII is replaced by a I in the next column to the left. The process is continued to the left of the table. The normalized form of a number is unique. In a normalized regular 5-tally system, each column can contain only five possible patterns. Call these 0, 1, 2, 3, and 4.

	I	II	III	IIII
0	1	2	3	4

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Collections of these symbols in each column represent numbers. A normalized tally table has exactly one symbol in each column. Arithmetic uses direct addition and then normalization.

```
(0;1;1;4;0) + (0;2;3;2;0) = (0+0;1+2;1+3;4+2;0+0)
                          = (0;3;4;4+2;0)
```

This is more clear when laid out in a table:

```
    (0   ; 1   ; 1   ; 4   ; 0   )
 +  (0   ; 2   ; 3   ; 2   ; 0   )
 =  (0+0 ; 1+2 ; 1+3 ; 4+2 ; 0+0 )
 =  (0   ; 3   ; 4   ; 4+2 ; 0   )
```

The problem is the 4+2, fixed by making it 1 and 1 in the next column.

```
(0;3;4;4+2;0) = (0;3;4+1;1;0) = (0;3+1;0;1;0)
              = (0;4;0;1;0)
```

To prove correctness, refer to the tally this abbreviates.

Exercise 1-1 (medium)

Prove that the normalized form of a number is unique.

The process of addition can be formalized.

+	0	1	2	3	4
0	00	01	02	03	04
1	01	02	03	04	10
2	02	03	04	10	11
3	03	04	10	11	12
4	04	10	11	12	13

Each cell of the table shows how to reduce two digits in a column to one digit in the same column and one digit in the next to the left. When the rightmost column has six digits in it, five such actions will reduce it to a single digit. Then the action can be repeated on the next column to the left. Eventually the table is normalized.

Worked Exercise 1

Starting with this table:

0	1	3	4	1	2
0	3	2	0	4	2

Draw a line at the bottom under which to place the result, and a line at the top over which to place the shifted material.

The top row is an auxiliary row. It helps with the computation but is not part of the result.

						auxiliary
0	1	3	4	1	2	
0	3	2	0	4	2	
						result

