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Quantum fields in Minkowski spacetime

The theory of quantum fields in curved spacetime is a generalization of the well-established theory of quantum fields in Minkowski spacetime. To a great extent, the behavior of quantum fields in curved spacetime is a direct consequence of the corresponding flat spacetime theory. Local entities, such as the field equations and commutation relations, are to a large extent determined by the principle of general covariance and the principle of equivalence. However, global entities which are unique in Minkowski spacetime lose that uniqueness in curved spacetime. For example, the vacuum state, which in Minkowski spacetime is determined by Poincaré invariance, is not unambiguously determined in curved spacetime. This ambiguity is closely tied to the phenomenon of particle creation by certain gravitational fields, as in the expanding universe or near a black hole.

It is logical, therefore, to review the relevant aspects of flat spacetime quantum field theory. This will serve to establish the necessary background, to fix our notation, and to highlight those aspects of the theory which can be carried over to curved spacetime, as well as those which lose their meaning in curved spacetime. We will often be brief, emphasizing concepts while omitting many derivations, and only touch on particular topics. Our discussion of the curved spacetime theory in later chapters will be more detailed.

In this initial chapter, we discuss the canonical formulation, including the Schwinger action principle and the relation between symmetry transformations and conserved currents (Schwinger 1951b, 1953). We review the dynamical descriptions known as the Heisenberg picture, the Schrödinger picture, and the interaction picture. We introduce the Fock representations, in which states are described in terms of their particle content, and the Schrödinger representation, in which the states are described by field configurations. We include discussions of the Maxwell and Yang–Mills gauge fields, as well as the Dirac field, and the definitions of spin and angular momentum.

1.1 Canonical formulation

Recall that in classical mechanics the equations of motion of a particle or system of particles having independent generalized coordinates $q_i(t)$ and velocities $\dot{q}_i(t)$ are given by the principle of stationary action. This principle states that the action

$$S = \int_{t_1}^{t_2} dt\, L(q, \dot{q}) \tag{1.1}$$

is stationary under arbitrary variations of the q_i which vanish on the boundary of the region of integration. Here L is the Lagrangian of the system. The Hamiltonian is defined by

$$H(q,p) = \sum_i p_i \dot{q}_i - L,$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{1.2}$$

is the momentum conjugate to q_i. The system is quantized by taking the qs and ps to be Hermitian operators acting on a Hilbert space, and by imposing the canonical commutation relations

$$[q_i, q_j] = 0, \quad [p_i, p_j] = 0,$$

$$[q_i, p_j] = i\delta_{i,j}. \tag{1.3}$$

Here $\delta_{i,j}$ is the Kronecker delta. We are using units with $\hbar = c = 1$. From (1.3) it follows that if $F(q,p)$ is a function of the coordinate and momentum operators, then (assuming F can be Taylor expanded in p)

$$[q_i, F] = i\frac{\partial F}{\partial p_i}. \tag{1.4}$$

The above commutation relations imply that the q_i are a complete set of commuting observables with continuous spectra consisting (in the absence of impenetrable walls) of all real numbers. The same can be said for the p_i. (An observable is an Hermitian operator with a complete set of eigenstates.) The eigenstates[1] of the q_i are the kets $|q'>$, where q' denotes the set of eigenvalues q'_i of the operators q_i. Thus,

$$q_i|q'>= q'_i|q'>,$$

with the normalization

$$<q'|q''>= \delta(q' - q''),$$

[1] We use the conventions of Dirac (1958) in distinguishing the eigenvalue of an operator from the operator by using a $'$.

where $\delta(q' - q'')$ is the Dirac δ-function. It also follows that

$$< q'|p_i|q'' >= -i\frac{\partial\delta(q' - q'')}{\partial q'_i},$$

and that

$$< q'|p' >= (2\pi)^{-n/2} \exp\left(i\sum_{i=1}^{n} p'_i q'_i\right),$$

where $|p' >$ is an eigenket of the p_i having delta function normalization, and we have taken the index i to run from 1 to n. (For derivations, see Messiah (1961, pp. 302–309).) If $F(q, p)$ is a function of the q_i and p_i with any well-defined ordering of factors, then

$$< q'|F(q, p)|q'' >= F\left(q', -i\frac{\partial}{\partial q'}\right)\delta(q' - q''), \tag{1.5}$$

where $F(q', -i(\partial/\partial q'))$ is the same ordered function with $-i(\partial/\partial q'_i)$ replacing p_i in each position.

In the Schrödinger or configuration space representation, the abstract operators are represented by matrix elements based on the $|q' >$, such as that of p_i above, and the states are represented by functions. For example, a state $|\psi >$ is represented by the Schrödinger wave function $\psi(q') =< q'|\psi >$. An example is the wave function $< q'|p' >$ above, representing a particle of definite momentum. Similarly, in the momentum-space representation the operators are represented by matrix elements formed from the $|p' >$ and the states are represented by functions such as $< p'|\psi >$.

Up to now the description has been purely kinematical, with time playing no role. The dynamical evolution of the system is governed by the Hamiltonian $H(q, p, t)$. We have allowed for the possibility that H may have explicit time dependence, as through an interaction with an external field. The time evolution may be described in several physically equivalent ways, known as "pictures."

In the Schrödinger picture, the fundamental observables q and p do not change with time. Rather, the dynamical evolution of measurable quantities, such as expectation values of observables, is expressed through the time dependence of the ket describing the state of the system at each time. The fundamental dynamical equation is that of Schrödinger,

$$i\frac{d}{dt}|\psi(t) >= H(q, p, t)|\psi(t) >. \tag{1.6}$$

Because H may have explicitly time-dependent terms involving p and q, in general $H(t')$ and $H(t'')$ may not commute. (For brevity, we suppress the dependence of H on q and p.)

We note in passing that in the Schrödinger representation, the Schrödinger equation (1.6) becomes (using the completeness of the $|q' >$)

$$i < q'|\frac{d}{dt}|\psi(t) >= \int < q'|H(q,p,t)|q'' > dq'' < q''|\psi(t) >$$

or

$$i\frac{\partial}{\partial t}\psi(q',t) = H\left(q', -i\frac{\partial}{\partial q'}, t\right)\psi(q',t), \tag{1.7}$$

where $\psi(q',t) =< q'|\psi(t) >$.

The solution of (1.6) is

$$|\psi(t) >= U(t,t_0)|\psi(t_0) >,$$

with $U(t,t_0)$ satisfying

$$i\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0), \tag{1.8}$$

with the boundary condition

$$U(t_0,t_0) = 1.$$

The evolution operator $U(t,t_0)$ preserves the norm of the state vector and is thus unitary, satisfying

$$UU^\dagger = U^\dagger U.$$

From $U(t_0,t)U(t,t_0) = U(t_0,t_0) = 1$, it then follows that $U(t,t_0)^\dagger = U(t_0,t)$.

In the Heisenberg picture, the ket describing the state of the system does not change with time, while the dynamical evolution of the system is expressed through the time dependence of the fundamental observables $q(t)$ and $p(t)$. By applying $U(t,t_0)^\dagger$ to the Schrödinger picture ket describing the state of the system, we obtain a time-independent ket, which can be taken as the ket describing the state of the system in the Heisenberg picture. Thus, denoting quantities in the Schrödinger picture by subscript S and those in the Heisenberg picture by subscript H, we have

$$|\psi_H >= U(t,t_0)^\dagger|\psi_S(t) >= |\psi_S(t_0) > . \tag{1.9}$$

In order that measurable expectation values remain the same as in the Schrödinger picture, the Heisenberg picture operator $F_H(t)$ corresponding to a Schrödinger picture operator F_S must be related by

$$F_H(t) = U(t,t_0)^\dagger F_S U(t,t_0). \tag{1.10}$$

Note that the Hamiltonian H in (1.8) is H_S. When H_S has no explicit time dependence, the solution of (1.8) is

$$U(t,t_0) = \exp[-i(t-t_0)H],$$

and it follows that H commutes with U so that $H_H = H_S$. When H_S does have an explicit time dependence, $H_H \neq H_S$, and we must use (1.10) to define H_H in terms of H_S. In general, F_H will depend on time through $q_H(t)$, $p_H(t)$, and through further explicit appearance of t. Denoting by $\partial F_H/\partial t$ the derivative only with respect to this further explicit appearance of t, it follows from (1.8) and (1.10) that

$$i\frac{d}{dt}F_H = [F_H, H_H] + i\frac{\partial}{\partial t}F_H. \tag{1.11}$$

This resembles the classical equation of motion with the Poisson bracket replaced by $-i$ times the commutator. Of course, we do not need to define the Schrödinger picture before the Heisenberg picture. The latter is fully defined by stating that the ket describing the state of the system is independent of time and that operators F, constructed from the q_i, p_i (dropping subscript H), and t obey the Heisenberg equation of motion (1.11).

When two systems interact through a term in the Hamiltonian, which can be regarded as a perturbative interaction, it is useful to introduce another picture of the dynamical evolution, known as the "interaction picture." In this picture, the ket describing the state of the system evolves as in the Schrödinger picture, but only under the influence of the interaction term in the Hamiltonian, while operators evolve as in the Heisenberg representation, but only under the influence of the unperturbed term in the Hamiltonian.

Let us now turn to the canonical quantization of a system of independent real fields $\phi_a(x)$, where x refers to the Minkowski space and time coordinates x^μ, and the index a includes tensor or spinor indices and internal quantum numbers of the field multiplet. We will deal here with bosons and discuss later the modification of canonical quantization required with fermions. For brevity, the index a will often be suppressed; we can usually think of ϕ as a column or row matrix, depending on where it appears in an expression. Canonical quantization proceeds as in the previously discussed quantization of a particle. One thinks of the classical field $\phi(x)$ as analogous to the classical $q_i(t)$, with the spatial coordinates $\vec{x}$ regarded as labels like i. Because we are now dealing with a continuous label, Dirac δ-functions involving $\vec{x}$ will appear where Kronecker deltas involving the label i previously appeared. As before, we assume that the system is described by an action

$$S = \int_{t_1}^{t_2} dt\, L[\phi, \partial\phi]_t, \tag{1.12}$$

where the Lagrangian L is now a functional of the field ϕ and its first derivatives $\partial\phi/\partial x^\mu \equiv \partial_\mu\phi$, which are denoted collectively by $\partial\phi$. The subscript t indicates that L is a function of t. The Lagrangian L can be expressed in terms of a Lagrangian density $\mathcal{L}$ as

$$L[\phi, \partial\phi]_t = \int dV_x\, \mathcal{L}\big(\phi(\vec{x}, t), \partial\phi(\vec{x}, t)\big),$$

where dV_x is the spatial volume element. Because L is now a functional, the momentum π conjugate to the field $\phi(\vec{x},t)$ is defined in analogy with (1.2) through the following functional derivative (regarding $\partial_0\phi$ as independent of ϕ at time t),

$$\begin{aligned}\pi(\vec{x},t) &= \delta L[\phi,\partial\phi]_t/\delta\big(\partial_0\phi(\vec{x},t)\big)\\ &= \partial\mathcal{L}\big(\phi(\vec{x},t),\partial\phi(\vec{x},t)\big)/\partial\big(\partial_0\phi(\vec{x},t)\big).\end{aligned} \tag{1.13}$$

Here, we have used the definition of the functional derivative, which states that if $F[\chi]$ is a functional of $\chi(\vec{x})$, then under a variation $\delta\chi(\vec{x})$ of χ, which vanishes sufficiently fast at spatial infinity, we have

$$\delta F[\chi] = \int dV_x \,\frac{\delta}{\delta\chi(\vec{x})} F[\chi]\delta\chi(\vec{x}). \tag{1.14}$$

It follows that if the functional F has the form

$$F[\chi] = \int dV_x\, f\big(\chi(\vec{x}),\vec{\partial}\chi(\vec{x})\big),$$

then

$$\frac{\delta}{\delta\chi(\vec{x})}F[\chi] = \frac{\partial}{\partial\chi} f\big(\chi,\vec{\partial}\chi\big) - \partial_i\left(\frac{\partial}{\partial(\partial_i\chi)} f\big(\chi,\vec{\partial}\chi\big)\right),$$

where χ is evaluated at $\vec{x}$. The result in (1.13) follows from this with $\chi \to \partial_0\phi$. Another consequence is that

$$\frac{\delta\chi(\vec{x}\,')}{\delta\chi(\vec{x})} = \delta(\vec{x}\,' - \vec{x}).$$

One can regard the action in (1.12) as a functional depending on the space and time dependence of ϕ. Then, the Euler–Lagrange field equation below will be recognized as another application of the above result, but for a functional depending on one more dimension.

The Hamiltonian is defined by

$$H[\phi,\pi]_t = \int dV_x\, \pi_a(\vec{x},t)\partial_0\phi_a(\vec{x},t) - L[\phi,\partial\phi]_t. \tag{1.15}$$

(Although we write $H[\phi,\pi]$, dependence on spatial derivatives of ϕ (or π) is permitted.) The principle of stationary action yields upon variation of the fields in (1.12), the Euler–Lagrange field equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0, \tag{1.16}$$

where the repeated spacetime coordinate index μ is summed over its full range of values, in accordance with the Einstein summation convention.

The field is quantized in analogy with the canonical commutators of (1.3). Thus, we postulate that

$$[\phi_a(\vec{x},t),\phi_b(\vec{x}',t)]=0,\quad [\pi_a(\vec{x},t),\pi_b(\vec{x}',t)]=0,$$
$$[\phi_a(\vec{x},t),\pi_b(\vec{x}',t)]=i\delta_{a,b}\delta(\vec{x}-\vec{x}'),\tag{1.17}$$

where, as noted earlier, $\delta(\vec{x}-\vec{x}')$ is the Dirac δ-function. (In a theory of interacting fields, if we deal directly with the renormalized fields, then (1.17) is somewhat altered by a normalization factor. We are thus dealing here with the bare fields, which by definition satisfy the field equations with unrenormalized masses and coupling constants.) The t dependence is included in the commutation relations to emphasize that in a dynamical picture like the Heisenberg picture, in which the operators ϕ and π depend on time (as do the classical fields), the canonical commutation relations must be imposed on the fields and conjugate momenta evaluated at the same time. For it follows from (1.10) that $q_H(t)$ and $p_H(t)$ evaluated at the same time do satisfy (1.3), while that is not true in general if they are evaluated at different times. Of course, in the Schrödinger picture the fields and conjugate momenta have no time dependence, and t would not appear in (1.17).

The functional analogue of (1.4) follows from the above commutators:

$$[\phi_a(\vec{x}),F[\phi,\pi]\,]=i\frac{\delta}{\delta\pi_a(\vec{x})}F[\phi,\pi],\tag{1.18}$$

where we are suppressing the time dependence, taking all fields to be at a single time t. One can now set up a Schrödinger or field representation using the eigenstates of $\phi(\vec{x})$ defined by

$$\phi(\vec{x})|\phi'>=\phi'(\vec{x})|\phi'>.\tag{1.19}$$

The ket $|\phi'>$ corresponds to a state of the system in which the field has configuration $\phi'(\vec{x})$, where ϕ' is an ordinary or c-number function, unlike the field operator ϕ. Thus, we are using the analogue of the Dirac notation in which "eigenvalues" of the operator $\phi(\vec{x})$ are functions denoted by $\phi'(\vec{x})$. (Here the prime does not denote derivative, but instead distinguishes a c-number from an operator.) In ordinary quantum mechanics, it follows from (1.5) that if $|\psi>$ is an element of the Hilbert space spanned by the eigenkets $|q'>$, then

$$<q'|F(q,p)|\psi>=F\left(q',-i\frac{\partial}{\partial q'}\right)<q'|\psi>.$$

Similarly, we can show that if $|\Psi>$ is a state in the space spanned by the eigenstates $|\phi'>$, and $F[\phi,\pi]$ is a functional formed from the field operator and conjugate momentum, then

$$<\phi'|F[\phi,\pi]|\Psi>=F\left[\phi',-i\frac{\delta}{\delta\phi'}\right]<\phi'|\Psi>.\tag{1.20}$$

Here $<\phi'|\Psi>\equiv\Psi[\phi']$ is a complex number which is a functional of ϕ'. It is interpreted as the probability amplitude for finding the field observable ϕ to have the configuration or set of values given by $\phi'(\vec{x})$ when the system is in the state described by the vector $|\Psi>$.

If we work in the Schrödinger picture, then (1.20) can be used to turn the Schrödinger equation

$$i\frac{d}{dt}|\Psi(t)>= H[\phi,\pi]|\Psi(t)>$$

into the functional differential equation

$$i\frac{\partial}{\partial t}\Psi[\phi',t] = H\left[\phi', -i\frac{\delta}{\delta\phi'}\right]\Psi[\phi',t], \tag{1.21}$$

where $\Psi[\phi',t] \equiv <\phi'|\Psi(t)>$, and ϕ, π depend on $\vec{x}$, but not on t.

On the other hand, in the Heisenberg picture the state describing the evolving system is independent of time, while a general functional F of $\phi(\vec{x},t)$ and $\pi(\vec{x},t)$ will depend on time through its dependence on ϕ and π, as well as through a possible further explicit dependence on t. Then, the Heisenberg equation of motion is

$$i\frac{d}{dt}F[\phi,\pi;t] = \big[F[\phi,\pi;t],\, H[\phi,\pi]\big] + i\frac{\partial}{\partial t}F[\phi,\pi;t]. \tag{1.22}$$

The consistency of (1.22) with the Euler–Lagrange equations (1.16) can be proved in Minkowski spacetime and in the more general curved spacetime context (Parker, 1973, see Appendix B).

Finally, we expect that when there are no time- or space-dependent external parameters in the Lagrangian, then there should exist a conserved vector observable P^μ corresponding to the total energy and momentum of the system. For such a Lagrangian, by multiplying the Euler–Lagrange equations by $\partial_\mu\phi$ and using[2]

$$\partial_\mu\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\partial_\mu\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial\phi}\partial_\mu\phi,$$

we find immediately that

$$\partial_\mu T^\mu{}_\nu = 0, \tag{1.23}$$

where

$$T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L}. \tag{1.24}$$

Here, summation over internal indices a of ϕ and, when the fields are treated as operators, symmetrization over fields and their conjugate momenta are understood. The tensor $T_{\mu\nu}$ is called the canonical energy-momentum or stress tensor.

We mention in passing that in order to serve as the source in the Einstein gravitational field equations, $T_{\mu\nu}$ should be symmetric under interchange of indices.

[2] This assumes that the Lagrangian does not have any explicit dependence on the coordinates x^μ.

(This symmetry is also required if we wish to define a conserved angular momentum in terms of the energy-momentum tensor.) However, except in the case of particular forms of $\mathcal{L}$, the expression given in (1.24) (after lowering the index ν with the Minkowski metric $\eta_{\mu\nu}$) is not a symmetric tensor. A general manifestly symmetric expression for $T_{\mu\nu}$ will be given later, when we deal with curved spacetime. (Since both of these expressions for $T^{\mu}{}_{\nu}$ will satisfy (1.23) for *any* system with no explicit time or space dependence, we expect on physical grounds that they must each yield the same conserved energy and momentum P^{μ} to within a constant.) Furthermore, a modification of the canonical $T_{\mu\nu}$ that makes it symmetric, and yields the same P_{μ} and angular momentum as the original canonical $T_{\mu\nu}$, has been given by Belinfante (1939, 1940).

From (1.23) we have

$$\int dv_x\, \partial_\mu T^{\mu}{}_{\nu} = 0,$$

where dv_x denotes the *spacetime* volume element, and the integration is over a spacetime volume bounded by spatial infinity and any two constant-time hypersurfaces. Assuming that matrix elements of physical interest will be between states in which the physical field configuration is of finite spatial extent, we obtain by the Gauss divergence theorem the conservation law

$$\frac{d}{dt}\int dV_x\, T^{0}{}_{\nu} = 0,$$

where dV_x denotes the *spatial* volume element and the integration is over any constant-time hypersurface. Hence,

$$P_\nu = \int dV_x T^{0}{}_{\nu} \tag{1.25}$$

is the conserved energy-momentum vector. The sign in this definition is chosen so that $P_0 = H$, as can be verified by comparing (1.15) with (1.24).

As a special case of the Heisenberg field equation (1.22), suppose that the functional F is an ordinary function $f(\phi(x), \partial_i\phi(x), \pi(x))$ with no explicit time dependence. Then, (1.22) can be written as

$$i\partial_0 f = [f, P_0], \tag{1.26}$$

where the partial derivative symbol is used here in the conventional manner to indicate that the x^i are held fixed. In (1.22), the partial derivative symbol denoted derivation only with respect to explicit t dependence not coming from ϕ and π. The partial derivative in (1.26) includes *all* t dependence. The result in (1.26) is the 0-component of the more general relation

$$i\partial_\mu f = [f, P_\mu], \tag{1.27}$$

The $\mu = 0$ component, as noted above, follows from (1.22). For $\mu = i$, we easily verify (1.27) for powers of ϕ and π and thus for functions which can be expanded

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in power series in ϕ and π. For example, suppressing the t-dependence, which is the same in all arguments, we have

$$\begin{aligned}[\pi(\vec{x})^n, P_i] &= \int dV'_x\, \pi(\vec{x}')[\pi(\vec{x})^n, \partial'_i\phi(\vec{x}')] \\ &= -i\int dV'_x\, \pi(\vec{x}')n\pi(\vec{x})^{n-1}\partial'_i\delta(\vec{x}-\vec{x}') \\ &= i\partial_i(\pi(\vec{x})^n).\end{aligned}$$

Equation (1.27) also follows from a powerful generalization of the action principle and of Noether's theorem, known as the Schwinger operator action principle (Schwinger 1951b, 1953). (For textbook treatments see Roman (1969) or Toms (2007).) The action of (1.12) is integrated over a spacetime volume v bounded by two constant-time hypersurfaces at t_1 and t_2. (As originally stated, the principle deals with arbitrary spacelike hypersurfaces, but we work with constant-time hypersurfaces for simplicity at this stage. See Section 6.2 for a discussion.) Consider arbitrary infinitesimal variations, δx^μ and $\delta_0\phi(x)$, of the coordinates and field operators,

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \tag{1.28}$$

$$\phi(x) \to \phi'(x) = \phi(x) + \delta_0\phi(x), \tag{1.29}$$

where $\delta_0\phi(x)$ vanishes on the spatial boundary of integration at each time (i.e., it vanishes everywhere on the boundary of v, except on the interior of the constant-time hypersurfaces that bound v at t_1 and t_2). Then, the Schwinger action principle states that the variation of the action S of (1.12) has the form

$$\delta S = G(t_2) - G(t_1), \tag{1.30}$$

where the operator $G(t)$ is the generator of the above variation of the coordinates and fields at time t.

To say that G generates the variation means the following. For an operator functional $F[\phi, \pi]$, we have

$$i\delta_0 F = [F, G], \tag{1.31}$$

where all quantities are evaluated at the same time, and $\delta_0 F$ is the infinitesimal variation of F produced by (1.28) and (1.29). That is,

$$\delta_0 F = F[\phi + \delta_0\phi, \partial(\phi + \delta_0\phi)] - F[\phi, \partial\phi].$$

One can show from (1.30) that the generator has the form

$$G(t) = \int dV_x \left[\pi_a \delta\phi_a - T^0{}_\nu \delta x^\nu\right], \tag{1.32}$$

where $T^\mu{}_\nu$ is the energy-momentum tensor of (1.24), and

$$\delta\phi(x) \equiv \phi'(x') - \phi(x) \tag{1.33}$$