1 Introduction

Mechanics is the study of the behavior of matter under the action of internal and external forces. In this introductory treatment of continuum mechanics, we accept the concepts of time, space, matter, energy, and force as the Newtonian ideals. Here our objective is the formulation of engineering problems consistent with the fundamental principles of mechanics. To paraphrase Professor Y. C. Fung—there are generally two ways of approaching mechanics: One is the ad hoc method, in which specific problems are considered and specific solution methods are devised that incorporate simplifying assumptions, and the other is the general approach, in which the general features of a theory are explored and specific applications are considered at a later stage. Engineering students are familiar with the former approach from their experience with "Strength of Materials" in the undergraduate curriculum. The latter approach enables them to understand an entire field in a systematic way in a short time. It has been traditional, at least in the United States, to have a course in continuum mechanics at the senior or graduate level to unify the ad hoc concepts students have learned in the undergraduate courses. Having had the knowledge of thermodynamics, fluid dynamics, and strength of materials, at this stage, we look at the entire field in a unified way.

1.1 Concept of a Continuum

Although mechanics is a branch of physics in which, according to current developments, space and time may be discrete, in engineering the length and time scales are orders (and orders) of magnitude larger than those in quantum physics and we use space coordinates and time as continuous. The concept of a continuum refers to the treatment of matter as continuous. The justification for this, again, rests on the length scales involved. For example, consider a large volume V of air under constant pressure and temperature. Within this volume, visualize a small volume ΔV centered at a fixed point in space. Let us denote by ΔM the mass of material inside ΔV. The ratio $\Delta M/\Delta V$ is the average density ρ. However, if we shrink ΔV, we can imagine a state in which molecules of oxygen and nitrogen pass through it, and the concept of density loses its meaning. If the dimension of ΔV is kept large

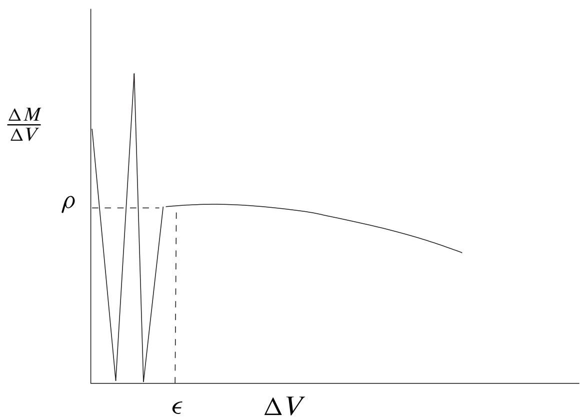

Figure 1.1. Density variation with volume.

compared with the mean free path of the molecules, we could use the concept of density. Mathematically, we say, "let $\Delta V \to 0$," but, physically, it is still kept above some value $\epsilon \gg 0$. We may also think of the situation as the discrete particles of matter are approximated as a continuous "smeared" state. Figure 1.1 shows a sketch of the limiting process.

To distinguish moving matter from its fixed-background space we use the term **material particle**, which is not to be confused with molecules or atoms.

1.2 Sequence of Topics

Prior to the consideration of mechanics topics such as stress and strain, the mathematical apparatus needed for our work is briefly reviewed. The fixed space in which the continuum moves is a three-dimensional (3D) Euclidean space. Cartesian tensors are essential for describing the deformation, motion, and the forces in mechanics within a Cartesian coordinate system. This topic is considered in Chapter 2. Although general tensors are not required for our studies, an understanding of this topic is worthwhile to appreciate its connection to geometry and mechanics. A number of advanced works in continuum mechanics use the general tensor formulation that is invariant under coordinate transformations involving curvilinear coordinates. Chapter 3 introduces some of the basic properties of general tensors.

Integral theorems of calculus, namely, the theorems of Gauss, Green, and Stokes, are extremely useful for our study of mechanics. These theorems are known to students from their studies of calculus. In Chapter 4 we visit these theorems by using index notation.

The next two chapters, Chapter 5 and Chapter 6, deal with the descriptions of the geometry of deforming bodies. Various strain measures and strain rate quantities are introduced in these chapters.

A separate chapter, Chapter 7, is devoted to a discussion of the fundamental axioms of mechanics. For a proper introduction of the stress tensor in Chapter 8, the fundamental axiom dealing with the balance of momentum is essential.

Thermodynamics plays an important role not only in restricting the form of stress–strain relations and stress–strain rate relations but also in explaining thermomechanical coupling. Chapter 9 refreshes the readers' knowledge in thermodynamics.

General forms of constitutive relations and their admissible forms are discussed in Chapter 10.

Chapter 11 considers elastic materials. It starts with nonlinear elastic materials, presents some of the classic inverse solutions, and then proceeds to linear elastic materials. A section on rubber elasticity is included because of the importance of this topic in engineering applications.

Chapter 12 deals with fluid dynamics. Again, classic inverse solutions of nonlinear fluids are presented first. Newtonian fluids and Navier–Stokes equations are briefly discussed.

Most students are unfamiliar with viscoelasticity and plasticity. These two topics are dealt with in Chapters 13 and 14. The treatment is of an elementary nature as this is assumed to be the first exposure of these two topics.

A digression into the available numerical solution techniques applicable to nonlinear problems is not made. The finite-element method has become the method of choice to deal with the solutions of solid mechanics problems. Finite-difference methods are often used in fluid dynamics problems. Other methods, such as molecular dynamics and Monte Carlo methods (see Frenkel and Smit, 2002), are actually outside the domain of continuum mechanics. However, these two methods play crucial roles in illuminating the foundations of irreversible thermodynamics (see Yourgrau, van der Merwe, and Raw, 1966) and continuum mechanics. Students are encouraged to supplement the topics covered here with courses in statistical mechanics (see Chandler, 1987) and related numerical methods.

SUGGESTED READING

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2 Cartesian Tensors

When the coordinates used to describe the geometry and deformation of a continuum and the forces involved are Cartesian, that is, three mutually orthogonal, right-handed coordinates with the Euclidean formula for distances, the quantities entering the equations of motion are conveniently described by use of the Cartesian tensors. Before we familiarize ourselves with these, let us examine a few related topics. These topics are included here primarily to establish our notation and to refresh the concepts the students might have seen in other contexts.

2.1 Index Notation and Summation Convention

Index notation uses coordinates x_1, x_2, and x_3 to denote the classical x, y, and z coordinates, respectively. The components of a vector $\boldsymbol{v}$ would be v_1, v_2, and v_3 (in three dimensions), instead of the conventional u, v, and w. As far as matrix elements are concerned, index notation, such as A_{23} to identify the element in the second row and third column, has been in use for some time. The advantage of index notation, in conjunction with the summation convention, is that we can shorten long mathematical expressions.

Consider a system of M equations, in N unknowns:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1N}x_N &= c_1, \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2N}x_N &= c_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &= ,\ldots, \\ A_{M1}x_1 + A_{M2}x_2 + \cdots + A_{MN}x_N &= c_M. \end{aligned} \tag{2.1}$$

This system of equations can also be written as

$$\sum_{j=1}^{N} A_{ij}x_j = c_i \quad (i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N).$$

In accordance with the **Einstein summation convention** we can further simplify the notation by writing

$$A_{ij}x_j = c_i \ (i = 1, 2, \ldots, M;\ j = 1, 2, \ldots, N), \tag{2.2}$$

where summation on the repeated index j is implied. Here, i is called a free index and j is called a dummy index. In dealing with three-dimensional (3D) Euclidean space, we have indices ranging from 1 to 3 (i.e., $M = N = 3$). Whenever an index is repeated once (and only once), the terms have to be summed with respect to that index. For this convention to be effective, extreme care must be taken to avoid the occurrence of any index more than twice. As examples, we have

$$A_{ii} = A_{11} + A_{22} + A_{33},$$

$$A_{ij}B_{ij} = B_{ij}A_{ij} = B_{ji}A_{ji}.$$

The symmetry of an array can be expressed as

$$A_{ij} = A_{ji}. \tag{2.3}$$

If an array is *skew symmetric* (also called antisymmetric),

$$B_{ij} = -B_{ji}. \tag{2.4}$$

An arbitrary array C can be expressed as the sum of a symmetric and a *skew-symmetric* array:

$$C_{ij} = A_{ij} + B_{ij}, \tag{2.5}$$

where

$$A_{ij} = C_{(ij)} = \frac{1}{2}(C_{ij} + C_{ji}), \quad B_{ij} = C_{[ij]} = \frac{1}{2}(C_{ij} - C_{ji}). \tag{2.6}$$

The subscripts inside the parentheses and square brackets help us to avoid introducing new variables A and B.

There are rare occasions when we would like to suppress the summation convention. Suppose we want to refer to A_{11}, A_{22}, or A_{33}; if we use A_{ii} we get the sum of the three terms. We may underline the repeated index to suppress the summation:

$$A_{\underline{ii}} = A_{11}, A_{22}, \text{or } A_{33}. \tag{2.7}$$

When we need to substitute one formula into another, we have to make sure that the dummy indices are distinct. For example,

$$a_i = C_{ij}b_j, \quad b_j = c_i D_{ji}. \tag{2.8}$$

A direct substitution for b_j in the first equation shows

$$a_i = C_{ij}c_i D_{ji}, \tag{2.9}$$

where i appears three times on the right-hand side, violating the summation rule. To avoid this, first we write

$$b_j = c_k D_{jk}, \tag{2.10}$$

and then substitute in the first equation to get

$$a_i = C_{ij}c_k D_{jk} = C_{ij}D_{jk}c_k, \tag{2.11}$$

where j and k are summation indices (or dummy indices) and the free index i appears on both sides. The dummy indices are similar to dummy variables used in integration of functions.

2.2 Kronecker Delta and Permutation Symbol

These two notations are extremely useful in connection with **Cartesian coordinates**. The Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}. \tag{2.12}$$

This definition assumes that i and j are explicit integers, such as $i = 1$ and $j = 3$, and it does not imply $\delta_{ii} = 1$. Elements of the Kronecker delta correspond to the elements of the identity matrix

$$\boldsymbol{I} = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.13}$$

With this,

$$\delta_{ii} = 3, \quad \delta_{ij}\delta_{jk} = \delta_{ik}, \quad \delta_{ij}A_{jk} = A_{ik}. \tag{2.14}$$

The permutation symbol or alternator is defined as

$$e_{ijk} = \begin{cases} 1, & \text{if} \quad i, j, k \text{ are even permutations of } 1, 2, 3 \\ -1, & \text{if} \quad i, j, k \text{ are odd permutations of } 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

From the preceding definition we have

$$e_{ijk} = e_{jki} = e_{kij} = -e_{ikj} = -e_{jik} = -e_{kji}, \tag{2.16}$$

$$e_{ijj} = e_{jij} = e_{jji} = 0. \tag{2.17}$$

Explicitly we have

$$e_{123} = e_{231} = e_{312} = 1, \quad e_{213} = e_{321} = e_{132} = -1. \tag{2.18}$$

Figure 2.1 shows the order of the indices for even and odd permutations.

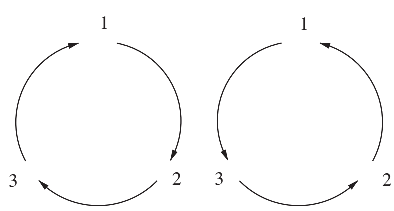

Figure 2.1. Even and odd permutations of the integers 1, 2, and 3.

Using the permutation symbol, we may express the determinant of a 3×3 – matrix $\boldsymbol{A}$ as

$$\begin{aligned}\det A = |\boldsymbol{A}| &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ &\quad - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \\ &= e_{ijk}A_{1i}A_{2j}A_{3k} \\ &= e_{ijk}A_{i1}A_{j2}A_{k3} \\ &= \tfrac{1}{6}e_{ijk}e_{\ell mn}A_{i\ell}A_{jm}A_{kn}.\end{aligned} \tag{2.19}$$

We can see the second and third equations as the (first) row expansion and the (first) column expansion of the determinant and the last equation as the sum of all row expansions and all column expansions (which add up to 6), divided by 6.

We can also express the determinant in terms of the cofactors of the matrix. For a 3×3 matrix the cofactor of an element A_{ij} is the 2×2 determinant we obtain by eliminating the ith row and jth column and multiplying it by $(-1)^{i+j}$. Let us denote this cofactor by A^*_{ij}. Then

$$|\boldsymbol{A}| = A_{ij}A^*_{ij} \quad \text{(no sum on } i\text{)}. \tag{2.20}$$

Observing that A^*_{ij} does not contain A_{ij} itself (recall, we eliminated a row and a column), we find

$$A^*_{ij} = \frac{\partial |A|}{\partial A_{ij}}. \tag{2.21}$$

When the inverse of the matrix exists, we have

$$A^{-1}_{ij} = \frac{A^*_{ji}}{|A|} = \frac{1}{|A|}\frac{\partial |A|}{\partial A_{ji}}. \tag{2.22}$$

A relation between the permutation symbols and the Kronecker deltas, known as the "e–δ identity," is useful in algebraic simplifications:

$$e_{ijk}e_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}. \tag{2.23}$$

From this, when $n = j$, we get

$$e_{ijk}e_{mjk} = 2\delta_{im}, \tag{2.24}$$

and, further, when $m = i$,

$$e_{ijk}e_{ijk} = 6. \tag{2.25}$$

2.2.1 Example: Skew Symmetry

If A_{ij} is a skew-symmetric matrix, solve the system of equations

$$e_{ijk}A_{jk} = B_i. \tag{2.26}$$

We use the "e–δ" identity to get

$$e_{imn}e_{ijk}A_{jk} = e_{imn}B_i,$$

$$(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})A_{jk} = e_{imn}B_i,$$

$$A_{mn} - A_{nm} = e_{imn}B_i,$$

$$A_{mn} = \frac{1}{2}e_{imn}B_i. \tag{2.27}$$

where we use the skew-symmetry property $A_{nm} = -A_{mn}$.

2.2.2 Example: Products

If A_{ij} is symmetric and B_{ij} is skew-symmetric, show that $A_{ij}B_{ij} = 0$.

Let

$$S = A_{ij}B_{ij}. \tag{2.28}$$

If we interchange the dummy indices $i \to j$ and $j \to i$, we have

$$S = A_{ji}B_{ji}. \tag{2.29}$$

Using the symmetry of A and the skew symmetry of B, we can write this as

$$S = A_{ij}(-B_{ij}) = -A_{ij}B_{ij} = -S, \quad 2S = 0, \quad S = 0. \tag{2.30}$$

2.3 Coordinate System

As shown in Fig. 2.2, we use a proper (right-handed) Cartesian coordinate system with x_1, x_2, and x_3 denoting the three axes. A directed line segment from the origin to any point in this 3D space is called a position vector $\boldsymbol{r}$, with components x_1, x_2,

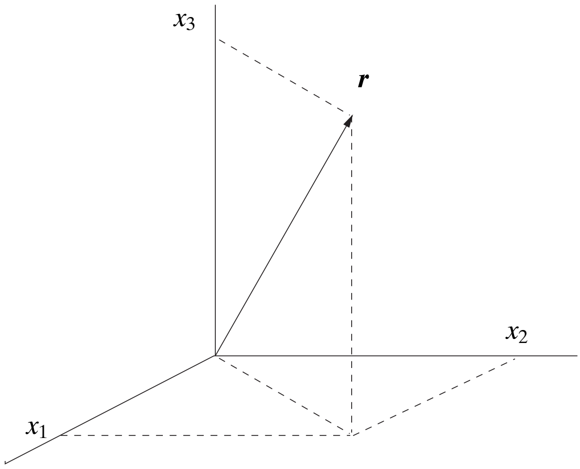

Figure 2.2. Cartesian coordinate system.

and x_3 along the three axes. We also use the notation $\boldsymbol{x}$ instead of $\boldsymbol{r}$ at times. In matrix notation, using a column vector, we have

$$\boldsymbol{r} = \boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}. \tag{2.31}$$

Defining unit vectors (or base vectors) $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ as

$$\boldsymbol{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \ \boldsymbol{e}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}, \ \boldsymbol{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \tag{2.32}$$

we can write

$$\boldsymbol{r} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3. \tag{2.33}$$

The **dot** and **cross products** of the unit vectors can be expressed with the Kronecker delta and the permutation symbol in the form

$$\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \delta_{ij}, \quad \boldsymbol{e_i} \times \boldsymbol{e_j} = e_{ijk}\boldsymbol{e_k}. \tag{2.34}$$

2.4 Coordinate Transformations

Two types of coordinate transformations are encountered frequently in our studies: coordinate translation and coordinate rotation. If we denote the new coordinates of a point $\boldsymbol{P}$ by x_i', in the case of translation, the two systems are related in the form

$$x_i' = x_i + h_i, \tag{2.35}$$

where h_i are constants. This is shown in Fig. 2.3.

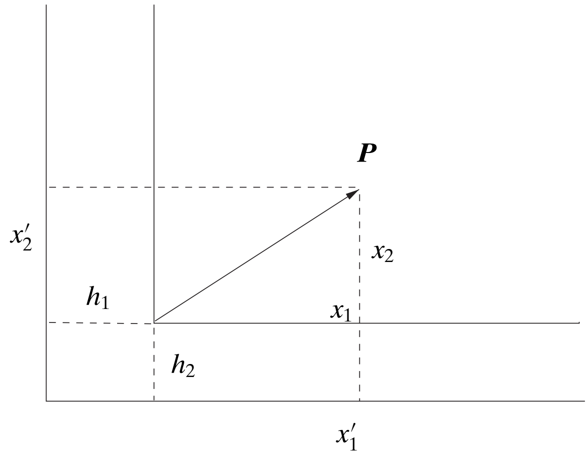

Figure 2.3. Coordinate translation.

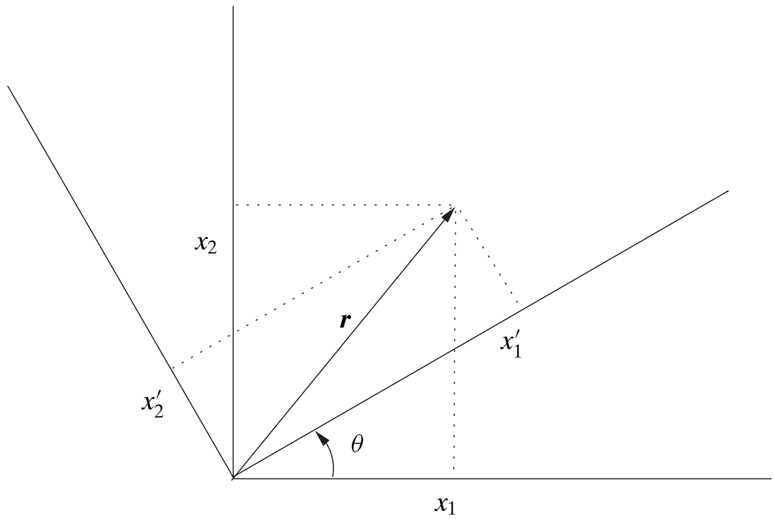

Figure 2.4. Coordinate rotation.

Let us consider first a rotation of coordinates in two dimensions. The origin remains fixed and we obtain the x_1' axis by rotating the x_1 axis counterclockwise by an angle, θ. As shown in Fig. 2.4, the two systems are related in the form

$$x_1' = x_1 \cos\theta + x_2 \sin\theta,$$
$$x_2' = -x_1 \sin\theta + x_2 \cos\theta. \tag{2.36}$$

We obtain the inverse of this transformation by replacing θ with $(-\theta)$, as

$$x_1 = x_1' \cos\theta - x_2' \sin\theta,$$
$$x_2 = x_1' \sin\theta + x_2' \cos\theta.$$

Equations (2.36) can also be written in matrix form:

$$\begin{Bmatrix} x_1' \\ x_2' \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \tag{2.37}$$

where

$$Q_{11} = Q_{22} = \cos\theta, \quad Q_{12} = -Q_{21} = \sin\theta. \tag{2.38}$$

Using column vectors $\boldsymbol{x}'$ and $\boldsymbol{x}$ and the square matrix $\boldsymbol{Q}$, we have

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x}. \tag{2.39}$$

A note of caution is in order at this point. We have absolute vectors in space, such as $\boldsymbol{r}$, and then we have column representations of the components in a chosen coordinate system, such as $\boldsymbol{x}$ and $\boldsymbol{x}'$. When there are no coordinate rotations involved, we do not have to distinguish these two representations. When there are coordinate rotations, we use $\boldsymbol{r}$ for the absolute vector.