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Introduction and theoretical background

The Earth's dynamical behaviour is a complex and fascinating subject with many practical ramifications. Its description requires the language of mathematics and computation. In this book, we attempt to make the theoretical foundations of the description of Earth's dynamics as complete as possible, and we accompany the theoretical descriptions with computer code and graphics for the implementation of the theory.

1.1 Scalar, vector and tensor analysis

We will make extensive use of scalars, vectors and tensors throughout the book. In this section, we will summarise the properties most often used. It is assumed that the reader is familiar with the elementary results of vector analysis summarised in Appendix A.

1.1.1 Scalars

Physical quantities determined by a single number such as mass, temperature and energy are *scalars*. Scalars are invariants under a change of co-ordinates, they remain the same in all co-ordinate systems. They are sometimes simply referred to as *invariants*. A scalar field is a function of space and time.

1.1.2 Vectors

Vectors require both magnitude and direction for their specification. They may be described by their components, their projections on the co-ordinate axes. An arbitrary vector then associates a scalar with each direction in space through an expression that is linear and homogeneous in the direction cosines.

In general, a vector can be defined in a space of arbitrary dimensions numbering two or greater. Our applications will be confined to a space of three dimensions and we will adopt this limitation. Let $\boldsymbol{r}$ be the radius vector to a point P specified by the three curvilinear co-ordinates u^1, u^2, u^3. Then $\boldsymbol{r}$ is described by the function

$$\boldsymbol{r} = \boldsymbol{r}(u^1, u^2, u^3). \tag{1.1}$$

The change in $\boldsymbol{r}$ due to differential displacements along the co-ordinate curves is then

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial u^1} du^1 + \frac{\partial \boldsymbol{r}}{\partial u^2} du^2 + \frac{\partial \boldsymbol{r}}{\partial u^3} du^3. \tag{1.2}$$

If two of the curvilinear co-ordinates are held fixed, the third describes a curve in space. Moving along u^j a unit distance from the point P, the change in $\boldsymbol{r}$ is equal to $\boldsymbol{b}_j = \partial \boldsymbol{r}/\partial u^j$. The *unitary vectors*

$$\boldsymbol{b}_1 = \frac{\partial \boldsymbol{r}}{\partial u^1}, \quad \boldsymbol{b}_2 = \frac{\partial \boldsymbol{r}}{\partial u^2}, \quad \boldsymbol{b}_3 = \frac{\partial \boldsymbol{r}}{\partial u^3}, \tag{1.3}$$

associated with the point P, form a base system for all vectors there. Any vector at that point can be expressed as a linear, homogeneous combination of the unitary base vectors. In particular,

$$d\boldsymbol{r} = \boldsymbol{b}_1 du^1 + \boldsymbol{b}_2 du^2 + \boldsymbol{b}_3 du^3. \tag{1.4}$$

The vector space, thus defined, may have different units measured along the co-ordinate directions. For instance, in thermodynamics the co-ordinates may represent pressure, volume and temperature, all in different units. This is an example of *affine geometry*. More commonly, we will be concerned with vectors in *metric geometry*, where the unitary vectors can be referred to a common unit of length. This allows measurement of the absolute value of a vector of arbitrary orientation and the distance between neighbouring points.

The three unitary base vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ define a parallelepiped with volume

$$V = \boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3) = \boldsymbol{b}_2 \cdot (\boldsymbol{b}_3 \times \boldsymbol{b}_1) = \boldsymbol{b}_3 \cdot (\boldsymbol{b}_1 \times \boldsymbol{b}_2), \tag{1.5}$$

using the properties (A.1) of the triple scalar product. A new triplet of base vectors, defined as

$$\boldsymbol{b}^1 = \frac{1}{V}(\boldsymbol{b}_2 \times \boldsymbol{b}_3), \quad \boldsymbol{b}^2 = \frac{1}{V}(\boldsymbol{b}_3 \times \boldsymbol{b}_1), \quad \boldsymbol{b}^3 = \frac{1}{V}(\boldsymbol{b}_1 \times \boldsymbol{b}_2), \tag{1.6}$$

are, in turn, orthogonal to the planes formed by the pairs $(\boldsymbol{b}_2 \times \boldsymbol{b}_3)$, $(\boldsymbol{b}_3 \times \boldsymbol{b}_1)$ and $(\boldsymbol{b}_1 \times \boldsymbol{b}_2)$. Adopting the *range convention*, whereby superscripts and subscripts are implied to range over the values 1, 2, 3, the two triplets of base vectors are found to obey

$$\boldsymbol{b}^i \cdot \boldsymbol{b}_j = \delta^i_j, \tag{1.7}$$

where δ^i_j is the *Kronecker delta*, which is equal to unity for $i = j$, zero otherwise. The original three unitary base vectors can be recovered from the new triplet of base vectors, using the properties (A.4) of the quadruple vector product, giving

$$\boldsymbol{b}_1 = V\left(\boldsymbol{b}^2 \times \boldsymbol{b}^3\right), \quad \boldsymbol{b}_2 = V\left(\boldsymbol{b}^3 \times \boldsymbol{b}^1\right), \quad \boldsymbol{b}_3 = V\left(\boldsymbol{b}^1 \times \boldsymbol{b}^2\right). \tag{1.8}$$

The triplet, $\boldsymbol{b}^1$, $\boldsymbol{b}^2$ and $\boldsymbol{b}^3$, are called *reciprocal unitary vectors*. They may be used as a base system of the vector space as an alternative to the original three unitary base vectors. In this base system, the differential $d\boldsymbol{r}$ becomes expressible as

$$d\boldsymbol{r} = \boldsymbol{b}^1 du_1 + \boldsymbol{b}^2 du_2 + \boldsymbol{b}^3 du_3. \tag{1.9}$$

Adopting the *summation convention*, whereby a repeated superscript or subscript implies summation over that index, equating the expressions (1.4) and (1.9) for the differential $d\boldsymbol{r}$ gives

$$d\boldsymbol{r} = \boldsymbol{b}_i\, du^i = \boldsymbol{b}^j du_j. \tag{1.10}$$

Taking the scalar product of this equation, first with $\boldsymbol{b}^k$, then with $\boldsymbol{b}_k$, and using the orthogonality relation (1.7), produces

$$du^k = \boldsymbol{b}^k \cdot \boldsymbol{b}^j du_j, \quad du_k = \boldsymbol{b}_k \cdot \boldsymbol{b}_i du^i. \tag{1.11}$$

Replacing the superscript k by i in the first relation, and replacing the subscript k by j in the second, relates the components of $d\boldsymbol{r}$ in the unitary and reciprocal unitary base systems by

$$du^i = g^{ij} du_j, \quad du_j = g_{ji} du^i, \tag{1.12}$$

with

$$g^{ij} = \boldsymbol{b}^i \cdot \boldsymbol{b}^j = g^{ji}, \quad g_{ji} = \boldsymbol{b}_j \cdot \boldsymbol{b}_i = g_{ij}. \tag{1.13}$$

An arbitrary vector $\boldsymbol{V}$ may be expressed as a linear combination of its components in the unitary base system $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, or in the reciprocal unitary base system $\boldsymbol{b}^1$, $\boldsymbol{b}^2$, $\boldsymbol{b}^3$, as

$$\boldsymbol{V} = v^i \boldsymbol{b}_i = v_j \boldsymbol{b}^j, \tag{1.14}$$

where, by (1.7), the components in each system are

$$v^i = \boldsymbol{V} \cdot \boldsymbol{b}^i, \quad v_j = \boldsymbol{V} \cdot \boldsymbol{b}_j. \tag{1.15}$$

Again, with scalar multiplication and using the orthogonality relation (1.7), the components in the unitary base system and in the reciprocal unitary base system are found to be related by

$$v^i = g^{ij} v_j, \quad v_j = g_{ji} v^i. \tag{1.16}$$

Replacing v^i and v_j in (1.14) by these expressions gives

$$\boldsymbol{V} = g^{ij} v_j \boldsymbol{b}_i = \boldsymbol{b}^i \cdot \boldsymbol{b}^j v_j \boldsymbol{b}_i = (\boldsymbol{b}^i \cdot \boldsymbol{V})\, \boldsymbol{b}_i \tag{1.17}$$

and

$$\boldsymbol{V} = g_{ji} v^i \boldsymbol{b}^j = \boldsymbol{b}_j \cdot \boldsymbol{b}_i v^i \boldsymbol{b}^j = (\boldsymbol{b}_j \cdot \boldsymbol{V})\, \boldsymbol{b}^j. \tag{1.18}$$

The components v^i are called the *contravariant components* of the vector $\boldsymbol{V}$ while the components v_j are called the *covariant components* of $\boldsymbol{V}$. By convention, contravariant components are indicated by a superscripted index and covariant components are indicated by a subscripted index.

Physical components of $\boldsymbol{V}$ can be defined by resolving the vector along a system of vectors of *unit* length. These may be defined as parallel to the unitary base system, with each reduced to unit length, by

$$\hat{\boldsymbol{e}}_1 = \frac{\boldsymbol{b}_1}{\sqrt{\boldsymbol{b}_1 \cdot \boldsymbol{b}_1}} = \frac{1}{\sqrt{g_{11}}}\boldsymbol{b}_1, \quad \hat{\boldsymbol{e}}_2 = \frac{1}{\sqrt{g_{22}}}\boldsymbol{b}_2, \quad \hat{\boldsymbol{e}}_3 = \frac{1}{\sqrt{g_{33}}}\boldsymbol{b}_3, \tag{1.19}$$

thus,

$$\boldsymbol{V} = V_1 \hat{\boldsymbol{e}}_1 + V_2 \hat{\boldsymbol{e}}_2 + V_3 \hat{\boldsymbol{e}}_3, \tag{1.20}$$

with physical components

$$V_1 = \sqrt{g_{11}}\, v^1, \quad V_2 = \sqrt{g_{22}}\, v^2, \quad V_3 = \sqrt{g_{33}}\, v^3. \tag{1.21}$$

Hence, the physical components, V_i, are of the same dimensions as the vector $\boldsymbol{V}$ itself.

We can now give dimensions to the vector space. The differential vector $d\boldsymbol{r}$ represents a displacement from the point P with co-ordinates (u^1, u^2, u^3) to the point with co-ordinates $(u^1 + du^1, u^2 + du^2, u^3 + du^3)$. Denoting the magnitude of this displacement by ds,

$$ds^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \boldsymbol{b}_i \cdot \boldsymbol{b}_j\, du^i\, du^j = \boldsymbol{b}^i \cdot \boldsymbol{b}^j\, du_i\, du_j, \tag{1.22}$$

or

$$ds^2 = g_{ij}\, du^i\, du^j = g^{ij}\, du_i\, du_j. \tag{1.23}$$

The coefficients g_{ij} and g^{ij} appear in bilinear forms expressing the square of the incremental displacement in terms of the increments in the co-ordinates u^i, or in terms of the increments in the reciprocal co-ordinates u_i. They are called *metrical coefficients*.

1.1.3 Vectors and co-ordinate transformations

Consider how the components of the vector $\boldsymbol{V}$ transform under a transformation of co-ordinates. Suppose we adopt a new set of curvilinear co-ordinates (u'^1, u'^2, u'^3) in place of (u^1, u^2, u^3). The unitary base vectors $\boldsymbol{b}_i$ are expressible as

$$\boldsymbol{b}_i = \frac{\partial \boldsymbol{r}}{\partial u^i} = \frac{\partial \boldsymbol{r}}{\partial u'^j}\frac{\partial u'^j}{\partial u^i} = \boldsymbol{b}'_j \frac{\partial u'^j}{\partial u^i}, \tag{1.24}$$

using the chain rule for partial derivatives. The vectors

$$\boldsymbol{b}'_j = \frac{\partial \boldsymbol{r}}{\partial u'^j} \tag{1.25}$$

form a new triplet of unitary base vectors in the new co-ordinate system. The vector $\boldsymbol{V}$ may be expressed by its contravariant components in either co-ordinate system as

$$\boldsymbol{V} = v^i \boldsymbol{b}_i = v'^j \boldsymbol{b}'_j. \tag{1.26}$$

Substituting for $\boldsymbol{b}_i$ from (1.24), we find that

$$\boldsymbol{V} = v^i \boldsymbol{b}_i = \frac{\partial u'^j}{\partial u^i} v^i \boldsymbol{b}'_j = v'^j \boldsymbol{b}'_j. \tag{1.27}$$

Thus, the transformation law for the contravariant components of the vector $\boldsymbol{V}$ is

$$v'^j = \frac{\partial u'^j}{\partial u^i} v^i. \tag{1.28}$$

The new covariant components of the vector $\boldsymbol{V}$ may be found from its new contravariant components, using the second of relations (1.16) and the metrical coefficient g'_{kj} in the new co-ordinate system, where

$$g'_{kj} = \boldsymbol{b}'_k \cdot \boldsymbol{b}'_j, \tag{1.29}$$

with $\boldsymbol{b}'_i$ being a unitary base vector in the new co-ordinate system. Using the chain rule for partial derivatives, the base vectors in the new co-ordinate system can be related to those in the original co-ordinate system by

$$\boldsymbol{b}'_k = \frac{\partial \boldsymbol{r}}{\partial u'^k} = \frac{\partial \boldsymbol{r}}{\partial u^l}\frac{\partial u^l}{\partial u'^k} = \boldsymbol{b}_l \frac{\partial u^l}{\partial u'^k}, \tag{1.30}$$

$$\boldsymbol{b}'_j = \frac{\partial \boldsymbol{r}}{\partial u'^j} = \frac{\partial \boldsymbol{r}}{\partial u^m}\frac{\partial u^m}{\partial u'^j} = \boldsymbol{b}_m \frac{\partial u^m}{\partial u'^j}. \tag{1.31}$$

Then,

$$g'_{kj} = \boldsymbol{b}'_k \cdot \boldsymbol{b}'_j = \boldsymbol{b}_l \cdot \boldsymbol{b}_m \frac{\partial u^l}{\partial u'^k}\frac{\partial u^m}{\partial u'^j} = \frac{\partial u^l}{\partial u'^k}\frac{\partial u^m}{\partial u'^j} g_{lm}. \tag{1.32}$$

Multiplying (1.28) through by g'_{kj} and summing over j, the new covariant components of $\boldsymbol{V}$ are found to be given by

$$v'_k = \frac{\partial u^l}{\partial u'^k}\frac{\partial u^m}{\partial u'^j}\frac{\partial u'^j}{\partial u^i} g_{lm} v^i, \tag{1.33}$$

where

$$\frac{\partial u^m}{\partial u'^j}\frac{\partial u'^j}{\partial u^i} = \frac{\partial u^m}{\partial u^i} = \delta^m_i. \tag{1.34}$$

Hence, (1.33) reduces to

$$v'_k = \frac{\partial u^l}{\partial u'^k}\delta^m_i g_{lm} v^i = \frac{\partial u^l}{\partial u'^k} v_l. \tag{1.35}$$

Thus, the transformation law for the covariant components of the vector $\boldsymbol{V}$ is

$$v'_j = \frac{\partial u^i}{\partial u'^j} v_i. \tag{1.36}$$

1.1.4 Tensors

In the analysis of the state of stress in a solid, it was realised that physical quantities more complicated than vectors were required for the description of the state of stress. Considering an imaginary surface within the stressed solid, with orientation described by its outward normal vector, the force on the surface is different for each orientation of the surface. Thus, the stress associates a force vector with each spatial direction. This has led to the definition of a *second-order tensor* as a physical quantity that associates a vector (*a first-order tensor*) with each spatial direction, and the generalisation that a tensor of order n associates a tensor of order $n - 1$ with each spatial direction.

The transformation laws, for contravariant vectors (1.28) and for covariant vectors (1.36), are easily generalised to those for tensors of second and higher order. In fact, we have already met a *second-order, twice covariant tensor*, the metrical coefficient g_{ij}, whose transformation law is given by (1.32) as

$$g'_{kl} = \frac{\partial u^i}{\partial u'^k}\frac{\partial u^j}{\partial u'^l} g_{ij}. \tag{1.37}$$

Replacing the index j by k in both of the relations (1.12), and i by j in the second, gives

$$du^i = g^{ik} du_k, \quad du_k = g_{kj} du^j. \tag{1.38}$$

Then,

$$du^i = g^{ik} du_k = g^{ik} g_{kj} du^j. \tag{1.39}$$

Comparing the left and right sides of this equation, we find that

$$g^{ik}g_{kj} = \delta^i_j. \tag{1.40}$$

Thus, the 3×3 matrix of components of g^{ik} multiplied by the 3×3 matrix of components of g_{kj} yields the unit matrix. Hence, the matrix representing g^{ik} is the inverse of the matrix representing g_{kj}. Since the inverse of a matrix is equal to the matrix of cofactors divided by the determinant, we have

$$g^{ik} = \frac{\Delta^{ik}}{g}, \tag{1.41}$$

where Δ^{ik} is the cofactor of the element g_{ik} of the 3×3 matrix of components of g_{kj} and g is the determinant given by

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}. \tag{1.42}$$

Symmetry of g^{ik} follows directly from the symmetry of g_{ik}. Now, suppose $\bar{g}^{ik}$ is a *second-order, twice contravariant tensor* with components equal to those of g^{ik} in one particular co-ordinate system. Then, by (1.40), in this co-ordinate system,

$$\bar{g}^{ik}g_{kj} = \delta^i_j. \tag{1.43}$$

This is now a tensor equation valid in all co-ordinate systems. In another system of co-ordinates it becomes

$$\bar{g}'^{ik}g'_{kj} = \delta^i_j. \tag{1.44}$$

Given g'_{kj}, we can take the inverse of the 3×3 matrix of its components as g'^{ik}, obeying

$$g'^{ik}g'_{kj} = \delta^i_j. \tag{1.45}$$

Comparing (1.44) and (1.45) it is found that

$$g'^{ik} = \bar{g}'^{ik} \tag{1.46}$$

in the new co-ordinate system as well, and thus they are identical in all co-ordinate systems. The metrical coefficient of the reciprocal unitary base system is then a *second-order, twice contravariant tensor* obeying the transformation law

$$g'^{kl} = \frac{\partial u'^k}{\partial u^i}\frac{\partial u'^l}{\partial u^j}g^{ij}. \tag{1.47}$$

The transformation laws (1.28), for first-order contravariant tensor components, and (1.36), for first-order covariant tensor components, apply to physical quantities more commonly known as vectors. The transformation laws (1.37) and (1.47) for the *metric tensors* are representative of those defining second-order, twice covariant and twice contravariant tensors. These transformation laws are easily generalised to define tensors of arbitrary order and even tensors of mixed contravariance and covariance.

When two tensors are multiplied and summed over one or more indices, such as in (1.43), the result is called the *contracted product*. The contracted product can be used to generate a *test for tensor character*. Suppose we have a quantity $t\,(i, j, k)$ with three indices i, j, k. Further suppose we have two contravariant vectors, $\boldsymbol{u}$ and $\boldsymbol{w}$, and a covariant vector, $\boldsymbol{v}$, and we form the triple contracted product

$$t\,(i, j, k)\, u^i v_j w^k = t^j_{ik}, \tag{1.48}$$

and find that it is invariant in all co-ordinate systems. Then t^j_{ik} is a third-order tensor, once contravariant in the index j, twice covariant in the indices i and k.

Such tests for tensor character can take many forms. For example, when we introduced the Kronecker delta in relation (1.7), we wrote it as though it were a second-order mixed tensor. If this were so, it would transform as

$$\delta'^{\,k}_{l} = \frac{\partial u'^{\,k}}{\partial u^i}\frac{\partial u^j}{\partial u'^{\,l}}\delta^i_j, \tag{1.49}$$

where $\delta'^{\,k}_{l}$ is the Kronecker delta in the new co-ordinate system. Taking the implied summation over j, the right side reduces to

$$\frac{\partial u'^{\,k}}{\partial u^i}\frac{\partial u^i}{\partial u'^{\,l}} = \frac{\partial u'^{\,k}}{\partial u'^{\,l}} = \delta'^{\,k}_{l}, \tag{1.50}$$

as required by the assumed transformation law (1.49). This identifies the Kronecker delta as a second-order mixed tensor, as assumed by our adopted notation.

1.1.5 Metric tensors and elements of arc, surface and volume

An infinitesimal displacement $d\boldsymbol{s}_1$, along the u^1 direction from a point P with co-ordinates (u^1, u^2, u^3), from (1.4), is

$$d\boldsymbol{s}_1 = \boldsymbol{b}_1\, du^1. \tag{1.51}$$

The magnitude of this infinitesimal displacement is

$$ds_1 = |\boldsymbol{b}_1|\, du^1 = \sqrt{\boldsymbol{b}_1 \cdot \boldsymbol{b}_1}\, du^1 = \sqrt{g_{11}}\, du^1, \tag{1.52}$$

using the second of relations (1.13). Similarly,

$$d\boldsymbol{s}_2 = \boldsymbol{b}_2\, du^2, \quad d\boldsymbol{s}_3 = \boldsymbol{b}_3\, du^3, \tag{1.53}$$

and

$$ds_2 = \sqrt{g_{22}}\, du^2, \quad ds_3 = \sqrt{g_{33}}\, du^3, \tag{1.54}$$

represent infinitesimal displacements in the u^2 and u^3 directions, respectively.

Next, we consider an element of surface area da_1, on the surface defined by u^1 constant, contained by the parallelogram formed by the infinitesimal displacements $\boldsymbol{ds}_2$, $\boldsymbol{ds}_3$ along the directions u^2, u^3. The area of the parallelogram is

$$da_1 = |\boldsymbol{ds}_2 \times \boldsymbol{ds}_3| = |\boldsymbol{b}_2 \times \boldsymbol{b}_3|\, du^2 du^3, \tag{1.55}$$

with

$$|\boldsymbol{b}_2 \times \boldsymbol{b}_3| = \sqrt{(\boldsymbol{b}_2 \times \boldsymbol{b}_3) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)}. \tag{1.56}$$

Using the vector identity (A.3),

$$(\boldsymbol{b}_2 \times \boldsymbol{b}_3) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3) = (\boldsymbol{b}_2 \cdot \boldsymbol{b}_2)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_3) - (\boldsymbol{b}_2 \cdot \boldsymbol{b}_3)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_2), \tag{1.57}$$

we find that

$$da_1 = \sqrt{g_{22}\, g_{33} - g_{23}^2}\, du^2 du^3. \tag{1.58}$$

Similarly, for elements on the u^2 and u^3 surfaces their areas are

$$da_2 = \sqrt{g_{33}\, g_{11} - g_{31}^2}\, du^3 du^1 \tag{1.59}$$

and

$$da_3 = \sqrt{g_{11}\, g_{22} - g_{12}^2}\, du^1 du^2. \tag{1.60}$$

An element of volume, bounded by the three co-ordinate surfaces, is given by

$$dv = \boldsymbol{ds}_1 \cdot (\boldsymbol{ds}_2 \times \boldsymbol{ds}_3) = \boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)\, du^1 du^2 du^3. \tag{1.61}$$

If we set $\boldsymbol{V} = \boldsymbol{b}_2 \times \boldsymbol{b}_3$ in the expression (1.17), we get

$$\boldsymbol{b}_2 \times \boldsymbol{b}_3 = \left[\boldsymbol{b}^i \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)\right] \boldsymbol{b}_i. \tag{1.62}$$

Replacing $\boldsymbol{b}^i$ by its expressions (1.6) and taking the scalar product with $\boldsymbol{b}_1$, we have

$$\begin{aligned}\boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3) = \frac{\boldsymbol{b}_1}{\boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)} \cdot \Big[&(\boldsymbol{b}_2 \times \boldsymbol{b}_3) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)\, \boldsymbol{b}_1 + (\boldsymbol{b}_3 \times \boldsymbol{b}_1) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)\, \boldsymbol{b}_2 \\ &+ (\boldsymbol{b}_1 \times \boldsymbol{b}_2) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)\, \boldsymbol{b}_3 \Big].\end{aligned} \tag{1.63}$$

Permuting subscripts in the first factor on the left of the vector identity (1.57), two additional vector identities emerge,

$$(\boldsymbol{b}_3 \times \boldsymbol{b}_1) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3) = (\boldsymbol{b}_3 \cdot \boldsymbol{b}_2)(\boldsymbol{b}_1 \cdot \boldsymbol{b}_3) - (\boldsymbol{b}_3 \cdot \boldsymbol{b}_3)(\boldsymbol{b}_1 \cdot \boldsymbol{b}_2) \tag{1.64}$$

and

$$(\boldsymbol{b}_1 \times \boldsymbol{b}_2) \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3) = (\boldsymbol{b}_1 \cdot \boldsymbol{b}_2)(\boldsymbol{b}_2 \cdot \boldsymbol{b}_3) - (\boldsymbol{b}_1 \cdot \boldsymbol{b}_3)(\boldsymbol{b}_2 \cdot \boldsymbol{b}_2)\,. \tag{1.65}$$

Using the three vector identities, expression (1.63) can be reduced to

$$\begin{aligned}[\boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)]^2 &= (\boldsymbol{b}_1 \cdot \boldsymbol{b}_1)[(\boldsymbol{b}_2 \cdot \boldsymbol{b}_2)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_3) - (\boldsymbol{b}_2 \cdot \boldsymbol{b}_3)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_2)] \\ &\quad + (\boldsymbol{b}_1 \cdot \boldsymbol{b}_2)[(\boldsymbol{b}_2 \cdot \boldsymbol{b}_3)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_1) - (\boldsymbol{b}_2 \cdot \boldsymbol{b}_1)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_3)] \\ &\quad + (\boldsymbol{b}_1 \cdot \boldsymbol{b}_3)[(\boldsymbol{b}_2 \cdot \boldsymbol{b}_1)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_2) - (\boldsymbol{b}_2 \cdot \boldsymbol{b}_2)(\boldsymbol{b}_3 \cdot \boldsymbol{b}_1)] \\ &= \begin{vmatrix} \boldsymbol{b}_1 \cdot \boldsymbol{b}_1 & \boldsymbol{b}_1 \cdot \boldsymbol{b}_2 & \boldsymbol{b}_1 \cdot \boldsymbol{b}_3 \\ \boldsymbol{b}_2 \cdot \boldsymbol{b}_1 & \boldsymbol{b}_2 \cdot \boldsymbol{b}_2 & \boldsymbol{b}_2 \cdot \boldsymbol{b}_3 \\ \boldsymbol{b}_3 \cdot \boldsymbol{b}_1 & \boldsymbol{b}_3 \cdot \boldsymbol{b}_2 & \boldsymbol{b}_3 \cdot \boldsymbol{b}_3 \end{vmatrix}.\end{aligned} \tag{1.66}$$

Replacing the scalar products with components of the covariant metric tensor defined in the second of relations (1.13), the determinant in expression (1.66) is identical to that defined by (1.42). Hence,

$$[\boldsymbol{b}_1 \cdot (\boldsymbol{b}_2 \times \boldsymbol{b}_3)]^2 = V^2 = g. \tag{1.67}$$

Then, the expression (1.61) for the volume element becomes

$$dv = \sqrt{g}\, du^1 du^2 du^3. \tag{1.68}$$

1.1.6 The cross product and differential operators

The three unitary base vectors define a parallelepiped with volume V given by any of the three scalar triple products as expressed in (1.5). On taking the square root of (1.67) this volume becomes

$$V = \sqrt{g}. \tag{1.69}$$

Suppose that, in addition to the arbitrary vector $\boldsymbol{V}$, expressed by (1.14) as a linear combination of its contravariant components and the unitary base vectors, we have a second arbitrary vector $\boldsymbol{W}$ similarly expressed. Then, the cross product of the two vectors is