PART ONE

DYNAMICS OF A SINGLE PARTICLE

1 Kinematics of a Particle

1.1 Introduction

One of the main goals of this book is to enable the reader to take a physical system, model it by using particles or rigid bodies, and then interpret the results of the model. For this to happen, the reader needs to be equipped with an array of tools and techniques, the cornerstone of which is to be able to precisely formulate the kinematics of a particle. Without this foundation in place, the future conclusions on which they are based either do not hold up or lack conviction.

Much of the material presented in this chapter will be repeatedly used throughout the book. We start the chapter with a discussion of coordinate systems for a particle moving in a three-dimensional space. This naturally leads us to a discussion of curvilinear coordinate systems. These systems encompass all of the familiar coordinate systems, and the material presented is useful in many other contexts. At the conclusion of our discussion of coordinate systems and its application to particle mechanics, you should be able to establish expressions for gradient and acceleration vectors in any coordinate system.

The other major topics of this chapter pertain to constraints on the motion of particles. In earlier dynamics courses, these topics are intimately related to judicious choices of coordinate systems to solve particle problems. For such problems, a constraint was usually imposed on the position vector of a particle. Here, we also discuss time-varying constraints on the velocity vector of the particle. Along with curvilinear coordinates, the topic of constraints is one most readers will not have seen before and for many they will hopefully constitute an interesting thread that winds its way through this book.

1.2 Reference Frames

To describe the kinematics of particles and rigid bodies, we presume on the existence of a space with a set of three mutually perpendicular axes that meet at a common point P. The set of axes and the point P constitute a reference frame. In Newtonian mechanics, we also assume the existence of an inertial reference frame. In this frame, the point P moves at a constant speed.

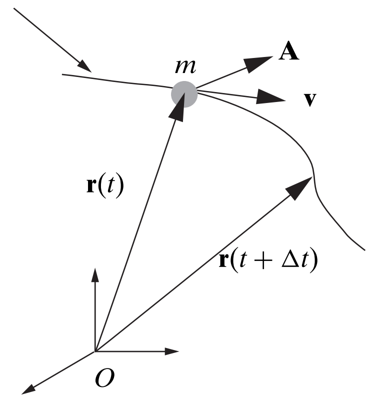


Figure 1.1. The path of a particle moving in $\mathbb{E}^3$. The position vector, velocity vector, and areal velocity vector of this particle at time t and the position vector of the particle at time $t + \Delta t$ are shown.

Depending on the application, it is often convenient to idealize the inertial reference frame. For example, for ballistics problems, the Earth's rotation and the translation of its center are ignored and one assumes that a point, say E, on the Earth's surface can be considered as fixed. The point E, along with three orthonormal vectors that are fixed to it (and the Earth), is then taken to approximate an inertial reference frame. This approximate inertial reference frame, however, is insufficient if we wish to explain the behavior of Foucault's famous pendulum experiment. In this experiment from 1851, Léon Foucault (1819–1868) ingeniously demonstrated the rotation of the Earth by using the motion of a pendulum.* To explain this experiment, it is sufficient to assume the existence of an inertial frame whose point P is at the fixed center of the rotating Earth and whose axes do not rotate with the Earth. As another example, when the motion of the Earth about the Sun is explained, it is standard to assume that the center S of the Sun is fixed and to choose P to be this point. The point S is then used to construct an inertial reference frame. Other applications in celestial mechanics might need to consider the location of the point P for the inertial reference frame as the center of mass of the solar system with the three fixed mutually perpendicular axes defined by use of certain fixed stars [80].

For the purposes of this text, we assume the existence of a fixed point O and a set of three mutually perpendicular axes that meet at this point (see Figure 1.1). The set of axes is chosen to be the basis vectors for a Cartesian coordinate system. Clearly, the axes and the point O are an inertial reference frame. The space that this reference frame occupies is a three-dimensional space. Vectors can be defined in this space, and an inner product for these vectors is easy to construct with the dot product. As such, we refer to this space as a three-dimensional Euclidean space and we denote it by $\mathbb{E}^3$.

* Discussions of his experiment and their interpretation can be found in [62, 138, 207]. Among his other contributions [215], Foucault is also credited with introducing the term "gyroscope."

1.3 Kinematics of a Particle

Suppose a single particle of mass m is in motion in $\mathbb{E}^3$. The position vector of the particle relative to a fixed origin O is denoted by $\mathbf{r}$ (see Figure 1.1). In mechanics, this vector is usually considered to be a function of time t: $\mathbf{r} = \mathbf{r}(t)$.

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ vectors of the particle are defined to be the respective first and second time derivatives of the position vector:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}.$$

It is crucial to note that, because $\mathbf{r}$ is measured relative to a fixed origin, $\mathbf{v}$ and $\mathbf{a}$ are the absolute velocity and acceleration vectors. By definition, the velocity vector can be calculated from the following limit:

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

We also use an overdot to denote the time derivative: $\mathbf{v} = \dot{\mathbf{r}}$ and $\mathbf{a} = \ddot{\mathbf{r}}$.

Supplementary to the aforementioned kinematical quantities, we also have the linear momentum $\mathbf{G}$ of the particle:

$$\mathbf{G} = m\mathbf{v}.$$

Further, the angular momentum $\mathbf{H}_O$ of the particle relative to O is

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}.$$

As we now show, this vector is related to the areal velocity vector $\mathbf{A}$.

As used in celestial mechanics, the magnitude of the areal velocity vector is the rate at which the position vector $\mathbf{r}$ of the particle sweeps out an area about the fixed point O (see, e.g., Moulton [150]). To establish an expression for this vector, we consider the position vector of the particle at time t and $t + \Delta t$. Then, the area of the parallelogram defined by these vectors is $\|\mathbf{r}(t) \times \mathbf{r}(t + \Delta t)\|$ (see Figure 1.1). This is twice the area swept out by the particle during the interval Δt. Taking the limit of the vector $\frac{\mathbf{r}(t)\times\mathbf{r}(t+\Delta t)}{2\Delta t}$ as $\Delta t \to 0$ and using the fact that $\mathbf{r}(t) \times \mathbf{r}(t) = \mathbf{0}$, we arrive at an expression for the areal velocity vector $\mathbf{A}(t)$:

$$\begin{aligned}
\mathbf{A}(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t) \times \mathbf{r}(t + \Delta t)}{2\Delta t} \\
&= \frac{1}{2}\mathbf{r}(t) \times \left(\lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t)}{\Delta t} \right) \\
&= \frac{1}{2}\mathbf{r}(t) \times \left(\lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \right).
\end{aligned}$$

That is,

$$\mathbf{A} = \frac{1}{2}\mathbf{r} \times \mathbf{v}. \tag{1.1}$$

The vector **A** plays an important role in several mechanics problems in which either the angular momentum $\mathbf{H}_O$ is constant or a component of $\mathbf{H}_O$ is constant. Several other examples of its use are discussed in the exercises at the end of this chapter.

Finally, we recall the definition of the kinetic energy T of the particle:

$$T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v}.$$

The definitions of the kinematical quantities that have been introduced are independent of the coordinate system that is used for $\mathbb{E}^3$. In solving most problems, it is crucial to have expressions for momenta and energies in terms of the chosen coordinate system. It is to this issue that we now turn.

1.4 Frequently Used Coordinate Systems

Depending on the problem of interest, there are several suitable coordinate systems for $\mathbb{E}^3$. The most commonly used systems are Cartesian coordinates $\{x = x_1, y = x_2, z = x_3\}$, cylindrical polar coordinates $\{r, \theta, z\}$, and spherical polar coordinates $\{R, \phi, \theta\}$. All of these coordinate systems can be considered as specific examples of a curvilinear coordinate system $\{q^1, q^2, q^3\}$ for $\mathbb{E}^3$, which we will discuss later on in this chapter.

Cartesian Coordinate System

For the Cartesian coordinate system, a set of right–handed orthonormal vectors are defined: $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Given any vector **b** in $\mathbb{E}^3$, this vector has the representation

$$\mathbf{b} = \sum_{i=1}^{3} b_i \mathbf{E}_i.$$

For the position vector **r**, we also have

$$\mathbf{r} = \sum_{i=1}^{3} x_i \mathbf{E}_i,$$

where $\{x_1, x_2, x_3\}$ are the Cartesian coordinates of the particle. Because $\mathbf{E}_i$ are fixed in both magnitude and direction, their time derivatives are zero: $\dot{\mathbf{E}}_i = \mathbf{0}$.

Cylindrical Polar Coordinates

A cylindrical polar coordinate system $\{r, \theta, z\}$ can be defined by a Cartesian coordinate system as follows:

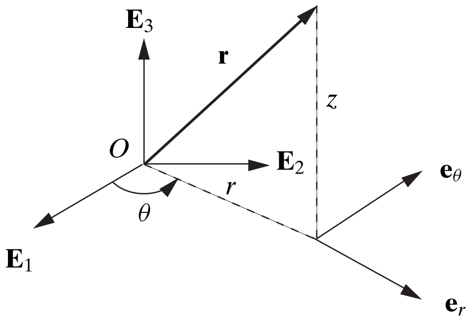
$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \qquad z = x_3,$$

where $\theta \in [0, 2\pi)$. Provided $r \neq 0$, then we can invert these relations to find that

$$x_1 = r\cos(\theta), \qquad x_2 = r\sin(\theta), \qquad x_3 = z.$$

In other words, given (x_1, x_2, x_3), a unique (r, θ, z) exists provided $(x_1, x_2) \neq (0, 0)$. Otherwise, when $r = 0$, the coordinate θ is ambiguous.

Figure 1.2. Cylindrical polar coordinates r, θ, and z.



Given a position vector $\mathbf{r}$, we can write

$$\begin{aligned}\mathbf{r} &= x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3 \\ &= r(\cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2) + z\mathbf{E}_3 \\ &= r\mathbf{e}_r + z\mathbf{E}_3,\end{aligned}$$

where, as shown in Figure 1.2, $\mathbf{e}_r = \cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2$.

It is convenient to define the set of unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$:

$$\mathbf{e}_r = \cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2, \qquad \mathbf{e}_\theta = \cos(\theta)\mathbf{E}_2 - \sin(\theta)\mathbf{E}_1, \qquad \mathbf{e}_z = \mathbf{E}_3.$$

We also notice that $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$, whereas $\dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$. We should also verify that $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{E}_z\}$ is a right-handed orthonormal basis for $\mathbb{E}^3$.*

Spherical Polar Coordinates

A spherical polar coordinate system $\{R, \phi, \theta\}$ can be defined by a Cartesian coordinate system as follows:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right), \qquad \phi = \tan^{-1}\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right),$$

where $\theta \in [0, 2\pi)$ and $\phi \in (0, \pi)$. Provided $\phi \neq 0$ or π, we can invert these relations to find

$$x_1 = R\cos(\theta)\sin(\phi), \qquad x_2 = R\sin(\theta)\sin(\phi), \qquad x_3 = R\cos(\phi).$$

Given a position vector $\mathbf{r}$, we can now write

$$\begin{aligned}\mathbf{r} &= x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + x_3\mathbf{E}_3 \\ &= R\sin(\phi)(\cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2) + R\cos(\phi)\mathbf{E}_3 \\ &= R\mathbf{e}_R,\end{aligned}$$

where, as shown in Figure 1.3, $\mathbf{e}_R = \sin(\phi)\cos(\theta)\mathbf{E}_1 + \sin(\phi)\sin(\theta)\mathbf{E}_2 + \cos(\phi)\mathbf{E}_3$.

* A basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is right-handed if $\mathbf{p}_3 \cdot (\mathbf{p}_1 \times \mathbf{p}_2) > 0$ and is orthonormal if the magnitude of each of the vectors $\mathbf{p}_i$ is 1 and they are mutually perpendicular: $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$, $\mathbf{p}_2 \cdot \mathbf{p}_3 = 0$, and $\mathbf{p}_1 \cdot \mathbf{p}_3 = 0$.

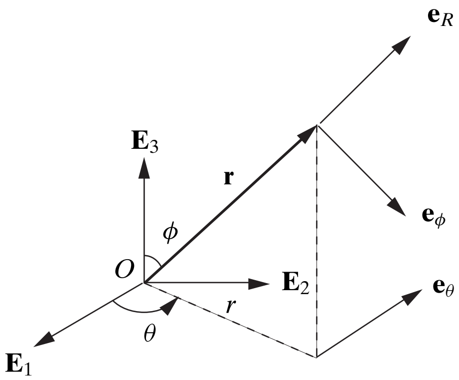


Figure 1.3. The spherical polar coordinates ϕ and θ.

For future purposes, it is convenient to define the right-handed orthonormal set of vectors $\{\mathbf{e}_R, \mathbf{e}_\phi, \mathbf{e}_\theta\}$:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) & \cos(\phi) \\ \cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

To establish the relations between these vectors and those defined earlier, we first calculate the intermediate relations

$$\begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{E}_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \sin(\phi) & 0 & \cos(\phi) \\ \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{E}_3 \end{bmatrix}. \tag{1.2}$$

These results enable us to transform among the three distinct sets of basis vectors.

As with the cylindrical polar coordinate system, the basis vectors we defined for the spherical polar coordinate system vary with the coordinates. Indeed, assuming that θ and ϕ are functions of time, a series of long calculations using (1.2) reveals that

$$\begin{bmatrix} \dot{\mathbf{e}}_R \\ \dot{\mathbf{e}}_\phi \\ \dot{\mathbf{e}}_\theta \end{bmatrix} = \begin{bmatrix} 0 & \dot{\phi} & \dot{\theta}\sin(\phi) \\ -\dot{\phi} & 0 & \dot{\theta}\cos(\phi) \\ -\dot{\theta}\sin(\phi) & -\dot{\theta}\cos(\phi) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix}. \tag{1.3}$$

These relations have an interesting form: Notice that the matrix in (1.3) is skew-symmetric. We shall see numerous examples of this later on when we discuss rotations and their time derivatives. Our later discussion should allow us to verify (1.3) rather easily.

1.5 Curvilinear Coordinates

The preceeding examples of coordinate systems can be considered as specific examples of a curvilinear coordinate system. The development of the vector calculus associated with such a system will be the focal point of this section of the book. Curvilinear coordinate systems have featured prominently in all areas of mechanics, and the material presented here has a wide range of applications. Most of our discussion is based on classical works and can be found in various textbooks on tensor calculus. Of these books, the one closest in spirit (and notation) to our treatment here is that of Simmonds [198]; [139, 201] are also recommended.

Consider a curvilinear coordinate system $\{q^1, q^2, q^3\}$ that is defined by the functions

$$
\begin{aligned}
q^1 &= \hat{q}^1\left(x_1, x_2, x_3\right), \\
q^2 &= \hat{q}^2\left(x_1, x_2, x_3\right), \\
q^3 &= \hat{q}^3\left(x_1, x_2, x_3\right).
\end{aligned}
\tag{1.4}
$$

We assume that the functions $\hat{q}^i$ are locally invertible:

$$
\begin{aligned}
x_1 &= \hat{x}_1\left(q^1, q^2, q^3\right), \\
x_2 &= \hat{x}_2\left(q^1, q^2, q^3\right), \\
x_3 &= \hat{x}_3\left(q^1, q^2, q^3\right).
\end{aligned}
\tag{1.5}
$$

This invertibility implies that, given the curvilinear coordinates of any point in $\mathbb{E}^3$, there is a unique set of Cartesian coordinates for this point and vice versa. Usually, the invertibility breaks down at several points in $\mathbb{E}^3$. For instance, the cylindrical polar coordinate θ is not uniquely defined when $x_1^2 + x_2^2 = 0$. This set of points corresponds to the x_3 axis.

Assuming invertibility, and fixing the value of one of the curvilinear coordinates, q^1 say, to equal q_0^1, we can determine the values of x_1, x_2, and x_3 such that the equation

$$
q_0^1 = \hat{q}^1\left(x_1, x_2, x_3\right)
$$

is satisfied. The union of all the points represented by these Cartesian coordinates defines a surface that is known as the q^1 coordinate surface (cf. Figure 1.4). If we move on this surface we find that the coordinates q^2 and q^3 will vary. Indeed, the curves on the q^1 coordinate surface that we find by varying q^2 while keeping q^3 fixed are known as q^2 coordinate curves.

More generally, the surface corresponding to a constant value of a coordinate q^j is known as a q^j coordinate surface. Similarly, the curve we obtain by varying the coordinate q^k while fixing the remaining two curvilinear coordinates is known as a q^k coordinate curve.

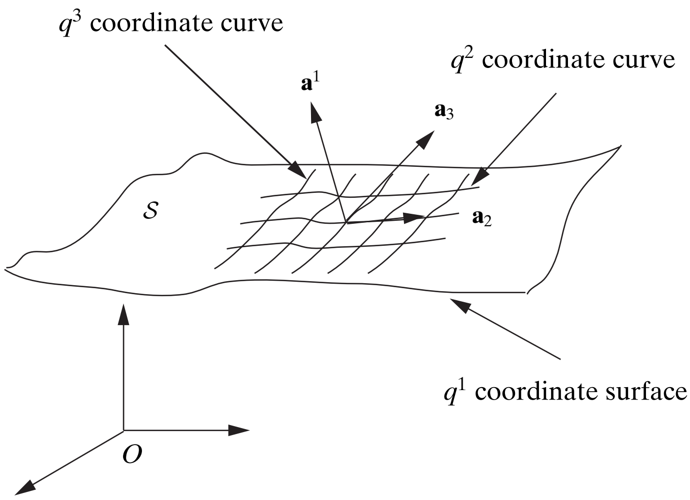


Figure 1.4. An example of a q^1 coordinate surface $\mathcal{S}$. At a point on this surface, $\mathbf{a}^1$ is normal to the surface, and $\mathbf{a}_2$ and $\mathbf{a}_3$ are tangent to the surface. The q^1 coordinate surface $\mathcal{S}$ is foliated by curves of constant q^2 and q^3.

Covariant Basis Vectors

Again assuming invertibility, we can express the position vector $\mathbf{r}$ of any point as a function of the curvilinear coordinates:

$$\mathbf{r} = \sum_{i=1}^{3} \hat{x}_i \left(q^1, q^2, q^3\right) \mathbf{E}_i.$$

It is also convenient to define the covariant basis vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$:

$$\begin{aligned} \mathbf{a}_i &= \frac{\partial \mathbf{r}}{\partial q^i} \\ &= \sum_{k=1}^{3} \frac{\partial \hat{x}_k}{\partial q^i} \mathbf{E}_k. \end{aligned}$$

Mathematically, when we take the derivative with respect to q^2 we fix q^1 and q^3; consequently, $\mathbf{a}_2$ points in the direction of increasing q^2. As a result, $\mathbf{a}_2$ is tangent to a q^2 coordinate curve. In general, $\mathbf{a}_i$ is tangent to a q^i coordinate curve.

You should notice that we can express the relationship between the covariant basis vectors and the Cartesian basis vectors in a matrix form:

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \frac{\partial \hat{x}_1}{\partial q^1} & \frac{\partial \hat{x}_2}{\partial q^1} & \frac{\partial \hat{x}_3}{\partial q^1} \\ \frac{\partial \hat{x}_1}{\partial q^2} & \frac{\partial \hat{x}_2}{\partial q^2} & \frac{\partial \hat{x}_3}{\partial q^2} \\ \frac{\partial \hat{x}_1}{\partial q^3} & \frac{\partial \hat{x}_2}{\partial q^3} & \frac{\partial \hat{x}_3}{\partial q^3} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

It is a good exercise to write out the matrix in the preceding equation for various examples of curvilinear coordinate systems, for instance, cylindrical polar coordinates.

Contravariant Basis Vectors

Curvilinear coordinate systems also have a second set of associated basis vectors: $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$. These vectors are known as the contravariant basis vectors. One method of defining them is as follows:

$$\mathbf{a}^1 = \sum_{i=1}^{3} \frac{\partial \hat{q}^1}{\partial x_i}\mathbf{E}_i, \qquad \mathbf{a}^2 = \sum_{i=1}^{3} \frac{\partial \hat{q}^2}{\partial x_i}\mathbf{E}_i, \qquad \mathbf{a}^3 = \sum_{i=1}^{3} \frac{\partial \hat{q}^3}{\partial x_i}\mathbf{E}_i.$$

That is,

$$\mathbf{a}^k = \nabla q^k.$$

Geometrically, $\mathbf{a}^i$ is normal to a q^i coordinate surface. However, as in the case of the covariant basis vectors, the contravariant basis vectors are not necessarily unit vectors, nor do they form an orthonormal basis for $\mathbb{E}^3$. Using the chain rule of calculus, we can show that

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta^i_j,$$

where δ^i_j is the Kronecker delta. As discussed in the Appendix, $\delta^i_j = 1$ if $i = j$ and is 0 otherwise. It is left as an exercise for the reader to show this result.*

Covariant and Contravariant Components

As $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\}$ form bases for $\mathbb{E}^3$, any vector $\mathbf{b}$ can be described as linear combinations of either sets of vectors:

$$\mathbf{b} = \sum_{i=1}^{3} b^i \mathbf{a}_i = \sum_{k=1}^{3} b_k \mathbf{a}^k.$$

The components b^i are known as the contravariant components, and the components b_k are known as the covariant components:

$$\mathbf{b}\cdot\mathbf{a}_i = \left(\sum_{k=1}^{3} b_k \mathbf{a}^k\right)\cdot\mathbf{a}_i = \sum_{k=1}^{3} b_k \delta^k_i = b_i,$$

$$\mathbf{b}\cdot\mathbf{a}^i = \left(\sum_{k=1}^{3} b^k \mathbf{a}_k\right)\cdot\mathbf{a}^i = \sum_{k=1}^{3} b^k \delta^i_k = b^i.$$

It is very important to note that $b^k \neq \mathbf{b}\cdot\mathbf{a}_k$ in general because $\mathbf{a}_i \cdot \mathbf{a}_k$ is not necessarily equal to δ^i_k.

The trivial case in which $x_i = q^i$ deserves particular mention. For this case, $\mathbf{r} = \sum_{k=1}^{3} x_i \mathbf{E}_i$. Consequently, $\mathbf{a}_i = \mathbf{E}_i$. In addition, $\mathbf{a}^i = \mathbf{E}_i$, and the covariant and contravariant basis vectors are equal.

* The starting point for this exercise is to note that $\frac{\partial x_k}{\partial x_j} = \delta^j_k$.