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978-0-521-87337-6 — Potential Flows of Viscous and Viscoelastic Liquids
Daniel Joseph , Toshio Funada , Jing Wang
Excerpt
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Introduction

The theory of potential flow is a topic in both the study of fluid mechanics and in mathematics. The mathematical theory treats properties of vector fields generated by gradients of a potential. The curl of a gradient vanishes. The local rotation of a vector field is proportional to its curl so that potential flows do not rotate as they deform. Potential flows are irrotational.

The mathematical theory of potentials goes back to the 18th century (see Kellogg, 1929). This elegant theory has given rise to jewels of mathematical analysis, such as the theory of a complex variable. It is a well-formed or "mature" theory, meaning that the best research results have already been obtained. We are not going to add to the mathematical theory; our contributions are to the fluid mechanics theory, focusing on effects of viscosity and viscoelasticity. Two centuries of research have focused exclusively on the motions of inviscid fluids. Among the 131,000,000 hits that come up under "potential flows" on Google search are mathematical studies of potential functions and studies of inviscid fluids. These studies can be extended to viscous fluids at small cost and great profit.

The fluid mechanics theory of potential flow goes back to Euler in 1761 (see Truesdell, 1954, §36). The concept of viscosity was not known in Euler's time. The fluids he studied were driven by pressures, not by viscous stresses. The effects of viscous stresses were introduced by Navier (1822) and Stokes (1845). Stokes (1851) considered potential flow of a viscous fluid in an approximate sense, but most later authors restrict their attention to "potential flow of an inviscid fluid." All the books on fluid mechanics and all courses in fluid mechanics have chapters on "potential flow of inviscid fluids" and none on the "potential flow of a viscous fluid."

An authoritative and readable exposition of irrotational flow theory and its applications can be found in chapter 6 of the book on fluid dynamics by Batchelor (1967). He speaks of the role of the theory of flow of an inviscid fluid:

> In this and the following chapter, various aspects of the flow of a fluid regarded as entirely inviscid (and incompressible) will be considered. The results presented are significant only inasmuch as they represent an approximation to the flow of a real fluid at large Reynolds number, and the limitations of each result must be regarded as important as the result itself.

In this book we consider irrotational flows of a viscous fluid. We are of the opinion that when one is considering irrotational solutions of the Navier–Stokes equations it is never necessary and typically not useful for one to put the viscosity to zero. This observation runs counter to the idea frequently expressed that potential flow is a topic that is useful only for inviscid fluids; many people think that the notion of a viscous potential flow is an oxymoron. Incorrect statements like "... irrotational flow implies inviscid flow but not the other way around" can be found in popular textbooks.

Irrotational flows of a viscous fluid scale with the Reynolds number as do rotational solutions of the Navier–Stokes equations generally. The solutions of the Navier–Stokes equations, rotational and irrotational, are thought to become independent of the Reynolds number at large Reynolds numbers. Unlike the theory of irrotational flows of inviscid fluids, the theory of irrotational flow of a viscous fluid can be considered as a description of flow at a finite Reynolds number.

Most of the classical theorems reviewed in chapter 6 of Batchelor's 1967 book do not require that the fluid be inviscid. These theorems are as true for viscous potential flow as they are for inviscid potential flow. Kelvin's minimum-energy theorem holds for the irrotational flow of a viscous fluid. The theory of the acceleration reaction leads to the concept of added mass; it follows from the analysis of unsteady irrotational flow. The theory applies to viscous and inviscid fluids alike.

It can be said that every theorem about potential flow of inviscid incompressible fluids applies equally to viscous fluids in regions of irrotational flow. Jeffreys (1928) derived an equation [his (20)] that replaces the circulation theorem of classical (inviscid) hydrodynamics. When the fluid is homogeneous, Jeffreys' equation may be written as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -\frac{\mu}{\rho}\oint \operatorname{curl}\boldsymbol{\omega}\cdot \mathrm{d}\mathbf{l}, \tag{1.0.1}$$

where

$$\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}, \quad C(t) = \oint \mathbf{u}\cdot\mathrm{d}\mathbf{l},$$

is the circulation around a closed material curve drawn in the fluid. This equation shows that

> ... the initial value of $\mathrm{d}C/\mathrm{d}t$ around a contour in a fluid originally moving irrotationally is zero, whether or not there is a moving solid within the contour. This at once provides an explanation of the equality of the circulation about an aeroplane and that about the vortex left behind when it starts; for the circulation about a large contour that has never been cut by the moving solid or its wake remains zero, and therefore the circulations about contours obtained by subdividing it must also add up to zero. It also indicates why the motion is in general nearly irrotational except close to a solid of to fluid that has pass near one.

Saint-Vénant (1869) interpreted the result of Lagrange (1781) about the invariance of circulation $\mathrm{d}C/\mathrm{d}t = 0$ to mean that

> vorticity cannot be generated in the interior of a viscous incompressible fluid, subject to conservative extraneous force, but is necessarily diffused inward from the boundaries.

Circulation formula (1.0.1) is an important result in the theory of irrotational flows of a viscous fluid. A particle that is initially irrotational will remain irrotational in motions that do not enter into the vortical layers at the boundary.

1.1 Irrotational flow, Laplace's equation

A potential flow is a velocity field $\mathbf{u}(\phi)$ given by the gradient of a potential ϕ:

$$\mathbf{u}(\phi) = \nabla\phi. \tag{1.1.1}$$

Potential flows have a zero curl:

$$\operatorname{curl}\mathbf{u} = \operatorname{curl}\nabla\phi = 0. \tag{1.1.2}$$

Fields that are curl free, satisfying (1.1.2), are called irrotational.

Vector fields satisfying the equation

$$\operatorname{div}\mathbf{u} = 0 \tag{1.1.3}$$

are said to be solenoidal. Solenoidal flows that are irrotational are harmonic; the potential satisfies Laplace's equation:

$$\operatorname{div}\mathbf{u}(\phi) = \operatorname{div}\nabla\phi = \nabla^2\phi = 0. \tag{1.1.4}$$

The theory of irrotational flow needed in this book is given in many books; for example, Lamb (1932), Milne-Thomson (1968), Batchelor (1967), and Landau and Lifshitz (1987). No-slip cannot be enforced in irrotational flow. However, eliminating all the irrotational effects of viscosity by putting $\mu = 0$ to reconcile our desire to satisfy no-slip at the cost of real physics is like throwing out the baby with the bathwater. This said, we can rely on the book by Batchelor and others for the results we need in our study of irrotational flow of viscous fluids.

1.2 Continuity equation, incompressible fluids, isochoric flow

The equation governing the evolution of the density ρ,

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\operatorname{div}\mathbf{u} = 0, \\ \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\nabla)\rho, \end{aligned} \tag{1.2.1}$$

is called the continuity equation. It guarantees that the mass of a fluid element is conserved.

If the fluid is incompressible, then ρ is constant and $\operatorname{div}\mathbf{u} = 0$. Flows of compressible fluids for which $\operatorname{div}\mathbf{u} = 0$ are called isochoric. Low-Mach-number flows are nearly isochoric. Incompressible and isochoric flows are solenoidal.

1.3 Euler's equations

Euler's equations of motion are given by

$$\begin{aligned} \rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla p + \rho\mathbf{g}, \\ \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}, \end{aligned} \tag{1.3.1}$$

where $\mathbf{g}$ is a body force per unit mass. If ρ is constant and $\mathbf{g} = \nabla G$ has a force potential, then

$$-\nabla p + \rho\mathbf{g} = -\nabla\hat{p}, \tag{1.3.2}$$

where $\hat{p} = p - \rho G$ can be called the pressure head. If $\mathbf{g}$ is gravity, then

$$G = \mathbf{g}\cdot\mathbf{x}.$$

To simplify the writing of equations we put the body force to zero except in cases for which it is important. If the fluid is compressible, then an additional relation, from thermodynamics, relating p to ρ is required. Such a relation can be found for isentropic flow or isothermal flow of a perfect gas. In such a system there are five unknowns, p, ρ, and $\mathbf{u}$, and five equations.

The effects of viscosity are absent in Euler's equations of motion. The effects of viscous stresses are absent in Euler's theory; the flows are driven by pressure. The Navier–Stokes equations reduce to Euler's equations when the fluid is inviscid.

1.4 Generation of vorticity in fluids governed by Euler's equations

Euler's equations (1.3.1) may be written as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} + \frac{1}{2}\nabla |\mathbf{u}|^2 = \mathbf{g} - \frac{\nabla p}{\rho}, \tag{1.4.1}$$

where we have used the vector identity

$$(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \frac{1}{2}\nabla |\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega} \tag{1.4.2}$$

and

$$\boldsymbol{\omega} = \boldsymbol{\omega}[\mathbf{u}] = \operatorname{curl}\mathbf{u}.$$

We can obtain the vorticity equation by forming the curl of (1.4.1):

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla\mathbf{u} - \operatorname{curl}\frac{\nabla p}{\rho} + \operatorname{curl}\mathbf{g}. \tag{1.4.3}$$

If $\operatorname{curl}(\nabla p/\rho) = \nabla \times [\frac{1}{\rho}p'(\rho)\nabla\rho] = 0$ and $\operatorname{curl}\mathbf{g} = 0$, then $\boldsymbol{\omega}[\mathbf{u}] = 0$; $\operatorname{curl}\mathbf{g} = 0$ if $\mathbf{g}$ is given by a potential $\mathbf{g} = \nabla G$. Flows for which $\operatorname{curl}(\nabla p/\rho) = 0$ are said to be barotropic. Barotropic flows governed by Euler's equations with conservative body forces $\mathbf{g} = \nabla G$ cannot generate vorticity. If the fluid is incompressible, then the flow is barotropic.

1.5 Perfect fluids, irrotational flow

Inviscid fluids that are also incompressible are called "perfect" or "ideal." Perfect fluids satisfy Euler's equations. Perfect fluids with conservative body forces give rise to Bernoulli's equation:

$$\nabla\left[\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right) + p - G\right] = 0,$$

where

$$\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2\right) + p - G = f(t). \tag{1.5.1}$$

We may absorb the function $f(t)$ into the potential $\hat{\phi} = \phi + \int^t f(t)\,\mathrm{d}t$ without changing the velocity:

$$\nabla\hat{\phi} = \nabla\left[\phi + \int^t f(t)\,\mathrm{d}t\right] = \nabla\phi = \mathbf{u}. \tag{1.5.2}$$

Bernoulli's equation relates p and ϕ before any problem is solved; actually curl $\mathbf{u} = 0$ is a constraint on solutions. The velocity $\mathbf{u}$ is determined by ϕ satisfying $\nabla^2\phi = 0$ and the boundary conditions.

1.6 Boundary conditions for irrotational flow

Irrotational flows have a potential, and if the flow is solenoidal the potential ϕ is harmonic, $\nabla^2\phi = 0$. This book reveals the essential role of harmonic functions in the flow of incompressible viscous and viscoelastic fluids. All the books on partial differential equations (PDEs) have sections devoted to the mathematical analysis of the Laplace's equation. Laplace's equation may be solved for prescribed data on the boundary of the flow region including infinity for flows on unbounded domains. It can be solved for Dirichlet data in which values of ϕ are prescribed on the boundary or for Neumann data in which the normal component of $\nabla\phi$ is prescribed on the boundary. Almost any combination of Dirichlet and Neumann data all over the boundary will lead to unique solutions of Laplace's equation.

It is important that unique solutions can be obtained when only one condition is prescribed for ϕ at each point on the boundary. The problem is overdetermined when two conditions are prescribed. We can solve the problem when Dirichlet conditions are prescribed or when Neumann conditions are prescribed but not when both are prescribed. In the case in which Neumann conditions are prescribed over the whole the boundary the solution is unique up to the addition of any constant. If the boundary condition is not specified at each point on the boundary, then the problem may be not overdetermined when two conditions are prescribed.

This point can be forcefully made within the framework of fluid dynamics. Consider, for example, an irrotational streaming flow over a body. The velocity $\mathbf{U}$ at infinity in the direction $\mathbf{x}$ is given by the potential $\phi = \mathbf{U} \cdot \mathbf{x}$. The tangential component of velocity on the boundary S of the body is given by

$$u_S = \mathbf{e}_S \cdot \mathbf{u} = \mathbf{e}_S \cdot \nabla\phi, \tag{1.6.1}$$

where $\mathbf{e}_S$ is a unit lying entirely in S. The normal component of velocity on S is given by

$$u_n = \mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \nabla\phi, \tag{1.6.2}$$

where $\mathbf{n}$ is the unit normal pointing from body to fluid.

Laplace's equation for streaming flow can be solved if

$$\phi \text{ is prescribed on } S, \tag{1.6.3}$$

or if

$$\mathbf{n} \cdot \nabla\phi \text{ is prescribed on } S. \tag{1.6.4}$$

It cannot be solved if ϕ and $\mathbf{n} \cdot \nabla\phi$ are simultaneously prescribed on S.

The prescription of the tangential velocity on S is a Dirichlet condition. If ϕ is prescribed on S the tangential derivatives can be computed. It is possible to solve Laplace's equation for a linear combination of ϕ and $\mathbf{n} \cdot \nabla\phi$; say $\mathbf{n} \cdot \nabla\phi + \alpha\phi$ is prescribed on S. This bind of condition is called a Robin boundary condition.

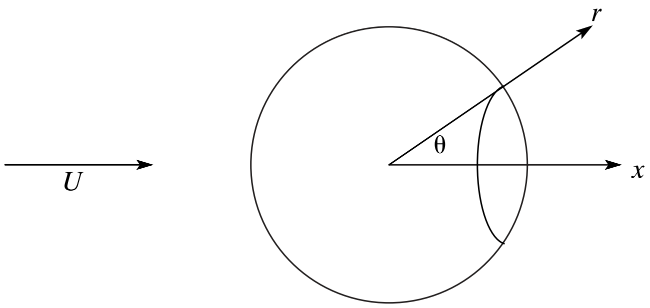

Figure 1.1. Axisymmetric flow over a sphere of radius a. The flow depends on the radius r and the polar angle θ.

The no-slip condition of viscous fluid mechanics requires that

$$u_S = 0, \quad u_n = 0 \tag{1.6.5}$$

simultaneously on S. These conditions cannot be satisfied by potential flow. In fact, these conditions cannot be satisfied by solutions of Euler's equations even when they are not irrotational.

The point of view that has been universally adopted by researchers and students of fluid mechanics for several centuries is that the normal component should be enforced so that the fluid does not penetrate the solid. The fluid must then slip at the boundary because there is no other choice. To reconcile this with no-slip, researchers put the viscosity to zero. The resolution of these difficulties lies in the fact that real flows have a nonzero rotational velocity at boundaries generated by the no-slip condition. The no-slip condition usually cannot be satisfied by the rotational velocity alone; the irrotational velocity is also needed (see chapter 4).

1.7 Streaming irrotational flow over a stationary sphere

A flow of speed U in the direction $x = r\cos\theta$ streams past a sphere of radius a (see figure 1.1). A solution ϕ of Laplace's equation $\nabla^2\phi = 0$ in spherical polar coordinates is

$$\phi = Ur\cos\theta + \frac{c}{r^2}\cos\theta. \tag{1.7.1}$$

The normal and tangential components of the velocity at $r = a$ are, respectively,

$$\frac{\partial\phi}{\partial r} = U\cos\theta - \frac{2c}{a^3}\cos\theta, \tag{1.7.2}$$

$$\frac{1}{a}\frac{\partial\phi}{\partial\theta} = -U\sin\theta - \frac{c}{a^3}\sin\theta. \tag{1.7.3}$$

If the normal velocity is prescribed to be zero on the sphere, then $c = Ua^3/2$ and

$$\phi_1 = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{1.7.4}$$

The corresponding tangential velocity is $-3U \sin\theta/2$. If the tangential velocity is prescribed to be zero on the sphere, then $c = -Ua^3$ and

$$\phi_2 = U\left(r - \frac{a^3}{r^2}\right)\cos\theta. \tag{1.7.5}$$

The corresponding normal velocity is $3U\cos\theta$.

The preceding analysis and uniqueness of solutions of Laplace's equation show that we may obtain the solution ϕ_1 with a zero normal velocity by prescribing the nonzero function $-\frac{3}{2}U\sin\theta$ for the tangential velocity. We may obtain the solution ϕ_2 with a zero tangential velocity by prescribing the nonzero function $3U\cos\theta$ for the normal velocity.

More complicated conditions for harmonic solutions on rigid bodies are encountered in exact solutions of the Navier–Stokes equations in which irrotational and rotational flows are tightly coupled at the boundary. The boundary conditions cannot be satisfied without irrotational flow and they cannot be satisfied by irrotational flow only [cf. equation (4.6.6)].

2

Historical notes

Potential flows of viscous fluids are an unconventional topic with a niche history assembled in a recent review article (Joseph, 2006a, 2006b).

2.1 Navier–Stokes equations

The history of Navier–Stokes equations begins with the 1822 memoir of Navier, who derived equations for homogeneous incompressible fluids from a molecular argument. Using similar arguments, Poisson (1829) derived the equations for a compressible fluid. The continuum derivation of the Navier–Stokes equation is due to Saint-Vénant (1843) and Stokes (1845). In his 1851 paper, Stokes wrote as follows:

> Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid (the equations (10) of my former (1845) paper), are
>
> $$\rho\left(\frac{\mathrm{D}u}{\mathrm{D}t} - X\right) + \frac{\mathrm{d}P_1}{\mathrm{d}x} + \frac{\mathrm{d}T_3}{\mathrm{d}y} + \frac{\mathrm{d}T_2}{\mathrm{d}z} = 0, \tag{132}$$
>
> with the two other equations which may be written down from symmetry. The pressures P_1, T_1, etc. are given by the equations
>
> $$P_1 = p - 2\mu\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \delta\right), \quad T_1 = -\mu\left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y}\right), \tag{133}$$
>
> and four other similar equations. In these equations
>
> $$3\delta = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}v}{\mathrm{d}y} + \frac{\mathrm{d}w}{\mathrm{d}z}. \tag{134}$$

The equations written by Stokes in his 1845 paper are the same ones we use today:

$$\rho\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \mathbf{X}\right) = \operatorname{div}\mathbf{T}, \tag{2.1.1}$$

where $\mathbf{X}$ is presumably a body force, which was not specified by Stokes, and

$$\mathbf{T} = \left(-p - \frac{2}{3}\mu \operatorname{div}\mathbf{u}\right)\mathbf{1} + 2\mu\mathbf{D}[\mathbf{u}], \tag{2.1.2}$$

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}, \tag{2.1.3}$$

$$\mathbf{D}[\mathbf{u}] = \frac{1}{2}\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T\right], \tag{2.1.4}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div}\mathbf{u} = 0. \tag{2.1.5}$$

Stokes assumed that the bulk viscosity $-\frac{2}{3}\mu$ is selected so that the deviatoric part of $\mathbf{T}$ vanishes and trace $\mathbf{T} = -3p$.

Inviscid fluids are fluids with zero viscosity. Viscous effects on the motion of fluids were not understood before the notion of viscosity was introduced by Navier in 1822. Perfect fluids, following the usage of Stokes and other 19th-century English mathematicians, are inviscid fluids that are also incompressible. Statements like Truesdell's (1954),

> In 1781 Lagrange presented his celebrated velocity-potential theorem: if a velocity potential exists at one time in a motion of an inviscid incompressible fluid, subject to conservative extraneous force, it exists at all past and future times.

though perfectly correct, could not have been asserted by Lagrange, because the concept of an inviscid fluid was not available in 1781.

2.2 Stokes theory of potential flow of viscous fluid

The theory of potential flow of a viscous fluid was introduced by Stokes in 1851. All of his work on this topic is framed in terms of the effects of viscosity on the attenuation of small-amplitude waves on a liquid–gas surface. Everything he said about this problem is subsequently cited. The problem treated by Stokes was solved exactly by Lamb (1932), who used the linearized Navier–Stokes equations, without assuming potential flow.

Stokes' discussion is divided into three parts, discussed in §§51, 52, and 53 of his (1851) paper:

(1) The dissipation method in which the decay of the energy of the wave is computed from the viscous dissipation integral in which the dissipation is evaluated on potential flow (§51).

(2) The observation that potential flows satisfy the Navier–Stokes equations together with the notion that certain viscous stresses must be applied at the gas–liquid surface to maintain the wave in permanent form (§52).

(3) The observation that, if the viscous stresses required for maintaining the irrotational motion are relaxed, the work of those stresses is supplied at the expense of the energy of the irrotational flow (§53).

Lighthill (1978) discussed Stokes' ideas but he did not contribute more to the theory of irrotational motions of a viscous fluid. On page 234 he notes that

> Stoke's ingenious idea was to recognise that the average value of this rate of working
>
> $$2\mu\left[(\partial\phi/\partial x)\,\partial^2\phi/\partial x\partial z + (\partial\phi/\partial z)\,\partial^2\phi/\partial z^2\right]_{z=0}$$
>
> required to maintain the unattenuated irrotational motions of sinusoidal waves must exactly balance the rate at which the same waves when propagating freely would lose energy by internal dissipation!.

Lamb (1932) gave an exact solution of the problem considered by Stokes in which vorticity and boundary layers are not neglected. Wang and Joseph (2006a) did purely irrotational theories of Stokes' problem that are in good agreement with Lamb's exact solution.

In fact, Stokes' idea called "ingenious" by Lighthill has serious defects in implementation. The unattenuated irrotational waves move with a speed independent of viscosity as would be true for waves on an inviscid fluid. Lamb's 1932 application of Stokes' idea gives rise to a good approximation of the decay rates due to viscosity for long waves but does not correct the wave speeds for the effects of viscosity. Moreover, the cutoff between long and short waves, which is defined by a condition (say large viscosity) for which the wave speed vanishes and progressive waves become standing waves, cannot be obtained from the dissipation calculation proposed by Stokes and implemented by Lamb and all other authors. A correct irrotational approximation, called VCVPF, which gives an approximation to Lamb's exact solution for all wavenumbers, including the dependence of the wave speed on viscosity and the all important cutoff value, was implemented by Wang and Joseph (2006b). Padrino and Joseph (2007) have demonstrated that all these results may be obtained by a revised and rigorous application of the dissipation method to the problem of viscous decay of capillary-gravity waves. They show that the effects of viscosity on the speed of progressive waves are sensible for waves near the cutoff value which are not too long and they obtain the irrotational approximation of this cutoff value (see figure 14.14).

2.3 The dissipation method

In his 1851 paper, Stokes writes:

> 51. By means of the expression given in Art. 49, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid. Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $u\,dx + v\,dy$ is an exact differential $d\phi$ when friction is neglected, and
>
> $$\phi = c\varepsilon^{-my}\sin(mx - nt)\,, \tag{140}$$
>
> where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation