1

Pp conditions

We start by introducing some elementary but possibly unfamiliar concepts: pp conditions and pp-types. These are used throughout the book. The link between pp conditions and functors is mentioned, though briefly; much more will be made of this later. The main theme of this chapter is the link between pp conditions and finitely presented modules. Free realisations of pp conditions are introduced. Elementary duality, which links pp conditions for right and left modules, is defined. This duality will be one of our main tools.

1.1 Pp conditions

After defining pp conditions and giving examples in Section 1.1.1 it is noted in Section 1.1.2 that these define functors from the category of modules to that of abelian groups. There is a potentially checkable criterion, 1.1.13, for implication of pp conditions, that is, for inclusion of the corresponding functors. In Section 1.1.3 we see that the set of pp conditions in a given number of free variables forms a modular lattice.

1.1.1 Pp-definable subgroups of modules

This section introduces pp (positive primitive) conditions and the sets they define in modules. The concept is illustrated by a variety of examples.

Consider a finite homogeneous system of R-linear equations:

$$\sum_{i=1}^{n} x_i r_{ij} = 0 \qquad j = 1, \dots, m.$$

Here the x_i are variables, the r_{ij} are elements of a given ring R and this is a system of equations for *right* R-modules.

We also write this as

$$\bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i r_{ij} = 0.$$

The symbol $\bigwedge$ is taken from mathematical logic and denotes conjunction, that is, "and". This system of equations may be regarded as a single condition, $\theta(\overline{x})$ say, on the tuple $\overline{x} = (x_1, \dots, x_n)$ of variables.

The **solution set** to $\theta(\overline{x})$ in any right R-module M is

$$\theta(M) = \left\{ \overline{a} = (a_1, \dots, a_n) \in M^n : \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_i r_{ij} = 0 \right\}.$$

This is an $\mathrm{End}(M)$-submodule of M^n, where the action of $\mathrm{End}(M)$ on M^n is the diagonal one: $f\overline{a} = (fa_1, \dots, fa_n)$ for $f \in \mathrm{End}(M)$ and $\overline{a} \in M^n$. It is not in general an R-submodule of M^n.

The simplest examples of such conditions θ are those of the form $xr = 0$ for some $r \in R$. In this case $\theta(M) = \{a \in M : ar = 0\} = \mathrm{ann}_M(r)$, the **annihilator** of r **in** M. Indeed any condition of the type above may be regarded as the generalised annihilator condition $\overline{x}H = 0$, where H is the $n \times m$ matrix $(r_{ij})_{ij}$. Then $\theta(M)$ is just the kernel of the map, $\overline{x} \mapsto \overline{x}H$ from M^n to M^m which is defined by right multiplication by the matrix H; a map of left $\mathrm{End}(M)$-modules.

Such annihilator-type conditions will not be enough: we close under projections to obtain generalised divisibility conditions.

Thus, given a subgroup $\theta(M)$ as defined above, consider its image under projection to the first k, say, coordinates:

$$\left\{ \overline{a} = (a_1, \dots, a_k) \in M^k : \exists\, a_{k+1}, \dots, a_n \in M \text{ such that } \bigwedge_{j=1}^{m} \sum_{i=1}^{n} a_i r_{ij} = 0 \right\}.$$

The condition $\phi(\overline{x}) = \phi(x_1, \dots, x_k)$, which is such that this projection of $\theta(M)$ is exactly its **solution set**, $\phi(M)$, in M, can be written

$$\exists x_{k+1}, \dots, x_n \bigwedge_{j=1}^{m} \sum_{i=1}^{n} x_i r_{ij} = 0.$$

This can be abbreviated as

$$\exists \overline{x}' \theta(\overline{x}, \overline{x}') \quad \text{where } \overline{x} = (x_1, \dots, x_k) \quad \text{and} \quad \overline{x}' = (x_{k+1}, \dots, x_n).$$

Any condition ϕ of this form is a **pp condition** and any subgroup of M^k of the form $\phi(M)$ is said to be a **pp-definable subgroup** of M or, more accurately, a **subgroup of M^k pp-definable** in M. The terminology "pp", an abbreviation of "positive primitive", is from logic and refers to the formal shape of the condition. The terms **finitely matrizable subgroup** and **subgroup of finite definition** are also used, following Zimmermann, respectively Gruson and Jensen, for what is here called a pp-definable subgroup. Note that $\phi(M)$ is a submodule of ${}_{\mathrm{End}(M)}M^k$, where $\mathrm{End}(M)$ has the diagonal action.

In the above condition ϕ the variables $x_1, \ldots, x_k$ are said to be **free** (to be replaced by values) whereas $x_{k+1}, \ldots, x_n$ are **bound** (by the existential quantifier). We write ϕ or $\phi(x_1, \ldots, x_k)$ depending on whether or not we wish to display the free variables. A condition like θ with no bound variables, that is, a system of linear equations, is a **quantifier-free** pp condition.

The simplest examples of pp conditions which are not annihilator conditions are divisibility conditions of the form $\exists y\,(x = yr)$. The solution set in M to this condition is $Mr = \{mr : m \in M\}$. Any pp condition can be expressed as a generalised divisibility condition: if $\phi(\overline{x})$ is $\exists \overline{y}\,(\overline{x}\;\overline{y})H = 0$, where $(\overline{x}\;\overline{y})$ should be read as the row vector with entries those of $\overline{x}$ followed by those of $\overline{y}$, then, writing H as a block matrix $H = \begin{pmatrix} A \\ -B \end{pmatrix}$, this condition may be rewritten as $\exists \overline{y}(\overline{x}A = \overline{y}B)$, and may be read as $B \mid \overline{x}A$, ("B divides $\overline{x}A$").

Examples 1.1.1. (a) Let $R = \mathbb{Z}$ and let p be a non-zero prime.

Let $M = \mathbb{Z}_p^{(I)}$ be an elementary abelian p-group, where the index set I may be infinite and where $\mathbb{Z}_p$ denotes the group with p elements. The condition $x = x$, which we write in preference to the equivalent $x0 = 0$, defines all of M and the condition $x = 0$ defines the zero subgroup. There are no more pp-definable subgroups because, for any two non-zero elements a, b of M there is $f \in \mathrm{End}(M)$ such that $fa = b$.

If, instead, we take $M = \mathbb{Z}_{p^n}^{(I)}$, for some n, then the pp-definable subgroups of M will be $M > Mp = \mathrm{ann}_M(p^{n-1}) > \cdots > Mp^{n-1} = \mathrm{ann}_M(p) > 0$. Clearly these are pp-definable, by both annihilation and divisibility conditions, and there are no more pp-definable subgroups because there are no more $\mathrm{End}(M)$-submodules.

Taking $M = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^3}$, again we obtain a chain

$$M > \mathbb{Z}_{p^2} \oplus p\mathbb{Z}_{p^3} > p\mathbb{Z}_{p^2} \oplus p^2\mathbb{Z}_{p^3} > p\mathbb{Z}_{p^2} \oplus p\mathbb{Z}_{p^3} > 0 \oplus p^2\mathbb{Z}_{p^3} > 0.$$

One may compute the cyclic $\mathrm{End}(M)$-submodules to see that there are no more pp-definable subgroups.[1]

Take $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ for an example where the pp-definable subgroups do not form a chain.

[1] In contrast to the erroneous diagram at [495, p. 22].

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In all those examples the pp-definable subgroups were exactly the $\mathrm{End}(M)$-submodules (4.4.25 says why) but that is not general nor even typical. For instance, a left coherent ring R which is not left noetherian has, by 2.3.19, left ideals which are not pp-definable as subgroups of the ring regarded as a right module, R_R, over itself.

(b) If R is any ring and $L = \sum_1^n Rr_i$ is a finitely generated left ideal of R, then L is a pp-definable subgroup of R_R. A defining condition for L is $\exists y_1, \ldots, y_n\, (x = \sum_1^n y_i r_i)$, for which we may alternatively use matrix notation,

$$\exists \overline{y}\, (x\ \overline{y}) \begin{pmatrix} 1 \\ -r_1 \\ \vdots \\ -r_n \end{pmatrix} = 0,$$

or "dot-product" notation, $\exists \overline{y}\, (x = \overline{y} \cdot \overline{r})$.

(c) Let $R = k\langle X, Y : YX - XY = 1\rangle$ be the first Weyl algebra over a field, k, of characteristic zero. Because R is a Dedekind (though not commutative) domain ([441, 5.2.11]), it is the case, see 2.4.10 and 2.4.15, that pp conditions for R-modules are simple combinations of basic annihilation and divisibility conditions.

Since k is in the centre of R, multiplication by any element of k on an R-module M is an R-endomorphism of M. Therefore every pp-definable subgroup of M is a vector space over k.

Every non-zero R-module is infinite-dimensional as a vector space over k: if M is finite-dimensional, with k-basis $a_1, \ldots, a_n$ say, then $\mathrm{ann}_R(M) = \bigcap_1^n \mathrm{ann}_R(a_i)$, is a co-finite-dimensional two-sided ideal of R so, since this ring is simple ([441, 1.3.5]), equals R. It will be shown, 8.2.28, that if M is a finitely generated module over this ring, then every pp-definable subgroup of M is either finite-dimensional or co-finite-dimensional. If, further, M is simple and $\mathrm{End}(M) = k$, for example, if M is simple and k is algebraically closed (see 8.2.27(3)), then every finite-dimensional subspace of M is pp-definable. But there will be co-finite-dimensional subspaces of M which are not pp-definable: for example, if k, hence R, is countable, then there are only countably many pp conditions but there are uncountably many co-finite-dimensional subspaces.

Similar remarks apply to Verma modules over $U(\mathrm{sl}_2(k))$, where k is an algebraically closed field of characteristic 0 (§8.2.4).

(d) Over a von Neumann regular ring every pp condition is equivalent to one which is quantifier-free, that is, to a system of linear equations, and this property characterises these rings (2.3.24).

It is easy to produce pp conditions more complicated than those above.

Example 1.1.2. Let k be a field and let R be the string algebra (Section 8.1.3) $k[a, b : ab = 0 = ba = a^2 = b^3]$, the k-path algebra of the Gelfand–Ponomarev quiver $\mathrm{GP}_{2,3}$ (p. 584). Take M to be the string module shown.

$$\begin{tikzcd}
& v_2 \arrow[dl, "a"'] \arrow[dr, "b"] & & v_4 \arrow[dl, "a"'] \arrow[dr, "b"] & & & \\
v_1 & & v_3 & & v_5 \arrow[dr, "b"] & & v_7 \arrow[dl, "a"'] \\
& & & & & v_6 &
\end{tikzcd}$$

Thus, as a k-vectorspace, M has basis $v_1, \dots, v_7$ and the actions of a and b are determined by their actions on the v_i, which are as shown. The convention is that the actions not shown are 0. So $v_7a = v_6$ and $v_7b = 0$.

There is a natural "pp-description" $\phi(x)$ of the element $v_1 \in M$, namely

$$xa = 0 \wedge \exists y\, (x = ya \wedge \exists z\, (yb = za \wedge \exists w\, (zb^2 = wa \wedge wb = 0))),$$

which rearranges to standard form as

$$\exists y, z, w\, (xa = 0 \wedge x = ya \wedge yb = za \wedge zb^2 = w \wedge wb = 0).$$

Clearly $v_1 \in \phi(M)$ but note that v_6 also satisfies the condition ϕ: directly or by 1.1.7, having noted that $v_2 \mapsto v_7$, $v_4 \mapsto 0$, $v_7 \mapsto 0$ defines an endomorphism of M taking v_1 to v_6. One may check that $\phi(M) = v_1k \oplus v_6k$. This subspace, which is $\ker(b^2) \cdot a$, can therefore be defined more simply by the pp condition $\exists y\, (x = ya \wedge yb^2 = 0)$ but one may produce arbitrarily complicated examples along these lines.

Example 1.1.3. Let Q be the quiver $\widetilde{D_4}$ which is as shown.

$$\begin{tikzcd}
1 \arrow[dr, "\alpha_1"] & & 2 \arrow[dl, "\alpha_2"] \\
& 0 & \\
4 \arrow[ur, "\alpha_4"] & & 3 \arrow[ul, "\alpha_3"']
\end{tikzcd}$$

Take R to be any ring and let M be an R-representation (Section 15.1.1) of $\widetilde{D_4}$. Thus M is given by replacing each vertex by an R-module and each arrow by an R-linear map; M may be regarded as a module over the path algebra $R\widetilde{D_4}$. The pp-definable subgroups of M include:

each "component" Me_i, where e_i is the idempotent of the path algebra corresponding to vertex i;
the image of each arrow $\mathrm{im}(\alpha_i) = M\alpha_i$ (α_i may be regarded as an element of $R\widetilde{D_4}$);
any R-submodule of Me_0 obtained from these four images *via* intersection and sum. There is quite a variety of these.

For example, suppose that k is a field, that V is a k-vector space and that f is an endomorphism of V. We can "code up" the structure (V, f) in a k-representation, M_f, of $\widetilde{D_4}$ as follows.

Set $V_0 = V \oplus V$ and let $\alpha_1, \dots, \alpha_4$ be, respectively, the inclusions of the following subspaces: $V \oplus 0$, $0 \oplus V$, $\Delta = \{(a, a) : a \in V\}$, $\mathrm{Gr}(f) = \{(a, fa) : a \in V\}$. Consider the following pp condition $\rho(x, y)$ with free variables x, y:

$$x = xe_1 \wedge y = ye_1 \wedge \exists u_2, u_4, u_3 \, (u_2 = u_2e_2 \wedge u_4 = u_4e_4 \wedge x\alpha_1 + u_2\alpha_2 \\ = u_4\alpha_4 \wedge u_3 = u_3e_3 \wedge y\alpha_1 + u_2\alpha_2 = u_3\alpha_3).$$

Unwinding this condition, one can see that $\rho(a, b)$ holds, that is, $(a, b) \in \rho(M)$, iff $a \in V$ (V identified with $V \oplus 0$) and $b = fa$.

Thus, for example, $\exists x \rho(x, y)$ and $\rho(x, 0)$ define, respectively, the image and kernel of f as subspaces of $V \oplus 0$. The pp condition $\rho_2(x, y)$ which is $\exists z \, (\rho(x, z) \wedge \rho(z, y))$ defines the graph of the action of f^2 on V, etc.

If, in place of $\widetilde{D_4}$ we were to take the "5-subspace quiver" which has an additional vertex and an arrow pointing from it to vertex 0, then we could code up[2] any additional endomorphism, g, of V via its graph as with f above. Thus pp conditions may be used to define the action of any (non-commutative) polynomial in f and g. Therefore the set of pp-definable subgroups is, in the case of this 5-subspace quiver, "wild".

Example 1.1.4. Let $\widetilde{A_1}$ be the quiver shown, the **Kronecker quiver**

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

and let M be any representation (over k). The subgroup, conveniently (and correctly if one thinks in terms of representations rather than modules) denoted $M\beta\alpha^{-1}$, that is, $\{a \in Me_1 : \alpha(a) \in \mathrm{im}(\beta)\}$, is pp-definable. Similarly $M\beta\alpha^{-1}\beta$, $M\beta\alpha^{-1}\beta\alpha^{-1}, \dots$ are pp-definable subgroups of M. For each $\lambda \in k$, so is the subgroup $\{a \in Me_1 : \beta(a) = \lambda\alpha(a)\}$.

[2] More precisely, interpret, in the sense of Section 18.2.1.

Example 1.1.5. Let D be a division ring. If M is any D-module, then $\mathrm{End}(M)$ acts transitively on the non-zero elements of M, so the only pp-definable subgroups of M are 0 and M.

Sometimes we will use the following notation from model theory: if χ is a condition with free variables $\overline{x}$, we write $\chi(\overline{x})$ if we wish to display these variables, and if $\overline{a}$ is a tuple of elements from the module M, then the notation $M \models \chi(\overline{a})$, read as "$M$ satisfies $\chi(\overline{a})$" or "$\overline{a}$ satisfies the condition χ in M", means $\overline{a} \in \chi(M)$, where $\chi(M)$ denotes the solution set of χ in M. We could dispense with this notation but much of the relevant literature makes use of it and we find it an efficient notation.[3]

Proposition 1.1.6. ([731, 6.5]) *Suppose that R is a local ring with (Jacobson) radical J and suppose that $J^2 = 0$. Then the pp-definable subgroups of R_R are R, J and the left ideals of finite length (that is, of finite dimension over the division ring R/J).*

Proof (in part). All the left ideals listed are pp-definable: if r is any non-zero element of J, then J is defined by $xr = 0$ and if $L \leq {}_R R$ is of finite length, generated by $r_1, \dots, r_n$ say, then L is defined by the pp condition $\exists y_1, \dots, y_n \, (x = \sum_1^n y_i r_i)$.

Suppose, for a contradiction, that some infinitely generated left ideal L were pp-definable, say $L = \phi(R)$. Zimmermann, see [731], deals with this by working directly with the matrices involved in the pp condition. Here we give an alternative proof in the case that the radical is split, i.e., $R = J + D \cdot 1$, where $D = R/J$. In that case if $(s_\lambda)_{\lambda \in \Lambda}$ is a D-basis for J, then $X_\lambda \mapsto s_\lambda$ induces an isomorphism $D[X_\lambda \, (\lambda \in \Lambda)] \, / \, \langle (X_\lambda X_\mu)_{\lambda,\mu} \rangle \simeq R$, where $\langle (X_\lambda X_\mu)_{\lambda,\mu} \rangle$ denotes the ideal generated by all the products $X_\lambda X_\mu$. Let σ be any automorphism of the D-vectorspace J and let $\alpha_\sigma : R \to R$ be the map induced by the endomorphism of the polynomial ring which takes X_λ to the linear combination of the X_μ with image σs_λ: clearly α_σ is an automorphism of the ring R. Applying α_σ to the elements of R appearing in the pp condition ϕ gives a pp condition, ϕ^σ say, and clearly $\phi^\sigma(R) = \alpha_\sigma(L)$.

Suppose now that R is countable. Since L is infinite-dimensional it has uncountably many images, $\alpha_\sigma(L)$, as σ varies. But, because R is countable, there are only countably many pp conditions ϕ^σ: a contradiction.

For the general case suppose that $\phi(x)$ is $\exists \overline{y} \, \theta(x, \overline{y})$, where θ is a quantifier-free pp condition (a system of linear equations). Choose a countable subset, T, of L which is linearly independent over D. For each $t \in T$ choose $\overline{s} = (s_1^t, \dots, s_n^t)$ from R such that $\theta(t, \overline{s})$ holds, that is, $(t, \overline{s}) \in \theta(R)$. Let $\Lambda' \subseteq \Lambda$ be such that all

[3] Under our notational conventions, writing $\chi(\overline{a})$ already implies that the length of $\overline{a}$ equals that of $\overline{x}$.

$t \in T$ and all corresponding s_i^t lie in the subring of R generated by 1 together with the s_λ with $\lambda \in \Lambda'$. If necessary put an additional λ with $X_\lambda \notin L$ into Λ' Let D' be a countable sub-division-ring of D such that every $d \in D$ appearing in ϕ, or in any expression of a $t \in T$ or an s_i^t as a linear combination of the s_λ, is in D'. Then $R' = D'[X_\lambda(\lambda \in \Lambda')]/\langle (X_\lambda X_\mu)_{\lambda,\mu} \rangle$ is a countable local ring with radical squared 0 and, by construction, $\phi(R') = L \cap R' < J'$, giving a contradiction.

There is generalisation of this at 4.4.15. □

1.1.2 The functor corresponding to a pp condition

In this section a criterion (1.1.13, 1.1.17) for one pp condition to be stronger than another is established.

Each pp condition ϕ defines a functor, F_ϕ, from the category, Mod-R, of right R-modules to the category, **Ab**, of abelian groups. On objects the action of F_ϕ is $M \mapsto \phi(M)$. If $f : M \to N$ is a morphism in Mod-R, then $F_\phi f$ is defined to be the restriction/corestriction of f to a map from $\phi(M)$ to $\phi(N)$; this is well defined by the following lemma.

Lemma 1.1.7. *If $f : M \to N$ is a morphism and ϕ is a pp condition, then $f(\phi(M)) \le \phi(N)$.*

Proof. The condition $\phi(\overline{x})$ has the form $\exists \overline{y}\, \theta(\overline{x}, \overline{y})$, where θ is a finite system of R-linear equations. Let $\overline{a} \in \phi(M)$, so there is a tuple $\overline{b}$ from M such that $(\overline{a}, \overline{b}) \in \theta(M)$. Note that f preserves solutions of R-linear equations: $f(\sum_i a_i r_i + \sum_k b_k s_k) = \sum_i f(a_i) r_i + \sum_k f(b_k) s_k$. So $(f\overline{a}, f\overline{b}) \in \theta(N)$, hence $f\overline{a} \in \phi(N)$, as required. □

If the pp condition ϕ has free variables $\overline{x} = (x_1, \dots, x_n)$, then, observing that $\phi(M) \le M^n$, we see that F_ϕ is a subfunctor of the nth power of the forgetful functor from Mod-R to **Ab**.

It is an important property of these pp-defined functors that they commute with direct limits, 1.2.31, as well as products, 1.2.3. The full story is given by 10.2.30 (and 18.1.19).

Corollary 1.1.8. *If ϕ is a pp condition and M is any module, then $\phi(M)$ is closed under the (diagonal) action of* End(M)*, that is, $\phi(M)$ is a submodule of $_{\mathrm{End}(M)}M^n$, where ϕ has n free variables.*

Corollary 1.1.9. *If ϕ is a pp condition with one free variable, then $\phi(R_R)$ is a left ideal of R.*

Corollary 1.1.10. *If M is any module and ϕ is a pp condition with n free variables, then $M \cdot \phi(R) \le \phi(M) \le M^n$.*

Proof. By $M \cdot \phi(R)$ is meant $\{\sum_{j=1}^m a_j \overline{r}_j : a_j \in M,\ \overline{r}_j \in \phi(R),\ m \ge 1\}$. This is generated by its subgroups $a \cdot \phi(R)$ for $a \in M$, and these are the images of $\phi(R)$ under the morphisms $(r_1, \dots, r_n) \mapsto (ar_1, \dots, ar_n)$ from R^n to M^n. □

Example 1.1.11. Suppose that C is a finitely presented module, say $C = \sum_1^n c_i R$ with defining relations $\sum_1^n c_i r_{ij} = 0$, $j = 1, \dots, m$. That is, there is an exact sequence $R^m \to R^n \to C \to 0$ where the map between the free modules is given by left multiplication, on column vectors, by the matrix $(r_{ij})_{ij}$.

Define θ to be the quantifier-free pp condition $\bigwedge_1^m \sum_1^n x_i r_{ij} = 0$. Then $F_\theta \simeq \mathrm{Hom}_R(C, -)$, as functors from Mod-$R$ to **Ab**. For, if M is any module, then a morphism $f : C \to M$ is determined by the images, $fc_1, \dots, fc_n$, of $c_1, \dots, c_n$, and the tuple $(fc_1, \dots, fc_n)$ satisfies all the equations $\sum_1^n x_i r_{ij} = 0$. Conversely, any n-tuple, $(a_1, \dots, a_n)$, of elements of M which satisfies θ determines, by sending c_i to a_i, a morphism from C to M. So the functors F_θ and $\mathrm{Hom}_R(C, -)$ (usually we write just $\mathrm{Hom}(C, -)$ or even $(C, -)$) agree on objects and it is easy to see that they agree on morphisms, hence are isomorphic by the natural[4] transformation $\eta : \mathrm{Hom}(C, -) \to F_\theta$ defined by $\eta_M : f \in \mathrm{Hom}(C, M) \mapsto (fc_1, \dots, fc_n) \in M^n$ at each module M.

Since θ was an arbitrary quantifier-free pp condition this shows that quantifier-free pp conditions correspond exactly to representable functors. A more precise statement is at 10.2.34.

Example 1.1.12. Let $R = \mathbb{Z}$. The torsion functor, $M \mapsto \tau M = \{a \in M : an = 0$ for some $n \in \mathbb{Z}, n \neq 0\}$ is an infinite sum of pp-defined subfunctors of the forgetful functor, namely the $M \mapsto \mathrm{ann}_M(n)$ for $n \in \mathbb{Z}, n \neq 0$, and plausibly is not itself defined by a single pp condition. Indeed, it is not isomorphic to a pp-defined functor: various proofs are possible; for example, if $\tau(-)$ were of the form $F_\phi(-)$, then the class of torsion groups would be definable (by the pair $(x = x)/\phi(x)$) in the sense of Section 3.4.1 but that is clearly in contradiction to 3.4.7.

The next result gives the condition, in terms of the defining matrices, for one pp condition to imply another and hence for two pp conditions to be equivalent in the sense of having identical solution sets in every module (that is, they define the same functor). Our approach follows [564, p. 124 ff.], see also [497, p. 187].

If ϕ and ψ are pp conditions such that $\psi(M) \le \phi(M)$ in every module M, then we say that ψ **implies** or **is stronger than** ϕ and write $\psi \to \phi$ or, more often,

[4] Once a generating tuple for C has been chosen.

$\psi \le \phi$ (reflecting the ordering of the solution sets). It will be seen later, 1.2.23, that it is enough to check this in every finitely presented module.

We write $\phi \equiv \psi$ and say that these pp conditions are **equivalent** if $\phi \ge \psi$ and $\phi \le \psi$. In all these definitions ϕ and ψ are assumed to have the same number of free variables.[5] By a **pp-pair** ϕ/ψ we mean a pair of pp conditions with $\phi \ge \psi$.

It was observed earlier that every pp condition with free variables $\overline{x}$ can be written in the form $B \mid \overline{x}A$, that is, $\exists \overline{y}\,(\overline{y}B = \overline{x}A)$, for some matrices A, B. The following implications between such divisibility conditions are immediate:

$B \mid \overline{x}A \Rightarrow BC \mid \overline{x}AC$ for any matrix C;
$B \mid \overline{x}A \Rightarrow B_0 \mid \overline{x}A$ if $B = B_1B_0$ for some matrices B_1, B_0;
$B \mid \overline{x}A \Rightarrow B \mid \overline{x}D$ if $A = A_0B + D$ for some matrices A_0, D.

A consequence of the proof of the next lemma is that every implication between pp conditions is obtained by a sequence (of length no more than three) of such implications.

Lemma 1.1.13. ([495, 8.10]) *Let $\phi(\overline{x})$, being $\exists \overline{y}\,(\overline{x}\,\overline{y})H_\phi = 0$, and $\psi(\overline{x})$, being $\exists \overline{z}\,(\overline{x}\,\overline{z})H_\psi = 0$, be pp conditions. Then $\psi \le \phi$ iff there are matrices $G = \begin{pmatrix} G' \\ G'' \end{pmatrix}$ and K such that $\begin{pmatrix} I & G' \\ 0 & G'' \end{pmatrix} H_\phi = H_\psi K$, where I is the $n \times n$ identity matrix, n being the length of $\overline{x}$, and 0 denotes a zero matrix with n columns.*

Proof. Suppose that ψ is $B' \mid \overline{x}A'$ and ϕ is $B \mid \overline{x}A$, so $H_\psi = \begin{pmatrix} A' \\ -B' \end{pmatrix}$ and $H_\phi = \begin{pmatrix} A \\ -B \end{pmatrix}$.

($\Leftarrow$) With an obvious notation, let us write the three types of immediate implication seen above as, respectively, $\begin{pmatrix} A \\ -B \end{pmatrix} \Rightarrow \begin{pmatrix} AC \\ -BC \end{pmatrix}$, $\begin{pmatrix} A \\ -B_1B_0 \end{pmatrix} \Rightarrow \begin{pmatrix} A \\ -B_0 \end{pmatrix}$, $\begin{pmatrix} A_0B + D \\ -B \end{pmatrix} \Rightarrow \begin{pmatrix} D \\ -B \end{pmatrix}$. Suppose that we have the matrix equation in the statement of the result: so $A - G'B = A'K$ and $-G''B = -B'K$. Then $\begin{pmatrix} A' \\ -B' \end{pmatrix} \Rightarrow \begin{pmatrix} A'K \\ -B'K \end{pmatrix}$ (first type) $= \begin{pmatrix} A - G'B \\ -G''B \end{pmatrix} \Rightarrow \begin{pmatrix} A - G'B \\ -B \end{pmatrix}$ (second type) $\Rightarrow \begin{pmatrix} A \\ -B \end{pmatrix}$ (third type), as required.

($\Rightarrow$) Suppose that the matrices A', B' are $n \times m$ and $l \times m$ respectively. Let M be the module freely generated by elements $x_1, \dots, x_n, y_1, \dots, y_l$ subject to the

[5] Rather, to be in the same number of free variables: see the footnote 3 of Chapter 3.