Cambridge University Press
978-0-521-87044-3 - An Introduction to Continuum Mechanics: With Applications
J. N. Reddy
Excerpt
More information

1 Introduction

I can live with doubt and uncertainty and not knowing. I think it is much more interesting to live not knowing than to have answers that might be wrong.

Richard Feynmann

What we need is not the will to believe but the will to find out.

Bertrand Russell

1.1 Continuum Mechanics

The subject of *mechanics* deals with the study of motion and forces in solids, liquids, and gases and the deformation or flow of these materials. In such a study, we make the simplifying assumption, for analysis purposes, that the matter is distributed continuously, without gaps or empty spaces (i.e., we disregard the molecular structure of matter). Such a hypothetical continuous matter is termed a *continuum*. In essence, in a continuum all quantities such as the density, displacements, velocities, stresses, and so on vary continuously so that their spatial derivatives exist and are continuous. The continuum assumption allows us to shrink an arbitrary volume of material to a point, in much the same way as we take the limit in defining a derivative, so that we can define quantities of interest at a point. For example, density (mass per unit volume) of a material at a point is defined as the ratio of the mass Δm of the material to a small volume ΔV surrounding the point in the limit that ΔV becomes a value ϵ^3, where ϵ is small compared with the mean distance between molecules

$$\rho = \lim_{\Delta V \to \epsilon^3} \frac{\Delta m}{\Delta V}. \tag{1.1.1}$$

In fact, we take the limit $\epsilon \to 0$. A mathematical study of mechanics of such an idealized continuum is called *continuum mechanics*.

The primary objectives of this book are (1) to study the conservation principles in mechanics of continua and formulate the equations that describe the motion and mechanical behavior of materials and (2) to present the applications of these equations to simple problems associated with flows of fluids, conduction of heat, and deformation of solid bodies. While the first of these objectives is an important topic, the reason for the formulation of the equations is to gain a quantitative understanding of the behavior of an engineering system. This quantitative understanding is useful in the design and manufacture of better products. Typical examples of engineering problems, which are sufficiently simple to cover in this book, are described below. At this stage of discussion, it is sufficient to rely on the reader's intuitive understanding of concepts or background from basic courses in fluid mechanics, heat transfer, and mechanics of materials about the meaning of the stress and strain and what constitutes viscosity, conductivity, modulus, and so on used in the example problems below. More precise definitions of these terms will be apparent in the chapters that follow.

PROBLEM 1 (SOLID MECHANICS)

We wish to design a diving board of given length L (which must enable the swimmer to gain enough momentum for the swimming exercise), fixed at one end and free at the other end (see Figure 1.1.1). The board is initially straight and horizontal and of uniform cross section. The design process consists of selecting the material (with Young's modulus E) and cross-sectional dimensions b and h such that the board carries the (moving) weight W of the swimmer. The design criteria are that the stresses developed do not exceed the allowable stress values and the deflection of the free end does not exceed a prespecified value δ. A preliminary design of such systems is often based on mechanics of materials equations. The final design involves the use of more sophisticated equations, such as the three-dimensional (3D) elasticity equations. The equations of elementary beam theory may be used to find a relation between the deflection δ of the free end in terms of the length L, cross-sectional dimensions b and h, Young's modulus E, and weight W [see Eq. (7.6.10)]:

$$\delta = \frac{4WL^3}{Ebh^3}\,. \tag{1.1.2}$$

Given δ (allowable deflection) and load W (maximum possible weight of a swimmer), one can select the material (Young's modulus, E) and dimensions L, b, and h (which must be restricted to the standard sizes fabricated by a manufacturer). In addition to the deflection criterion, one must also check if the board develops stresses that exceed the allowable stresses of the material selected. Analysis of pertinent equations provide the designer with alternatives to select the material and dimensions of the board so as to have a cost-effective but functionally reliable structure.

PROBLEM 2 (FLUID MECHANICS)

We wish to measure the viscosity μ of a lubricating oil used in rotating machinery to prevent the damage of the parts in contact. Viscosity, like Young's modulus of solid materials, is a material property that is useful in the calculation of shear stresses

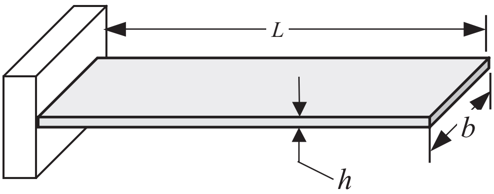

Figure 1.1.1. A diving board fixed at left end and free at right end.

developed between a fluid and solid body. A capillary tube is used to determine the viscosity of a fluid via the formula

$$\mu = \frac{\pi d^4}{128L} \frac{P_1 - P_2}{Q}, \tag{1.1.3}$$

where d is the internal diameter and L is the length of the capillary tube, P_1 and P_2 are the pressures at the two ends of the tube (oil flows from one end to the other, as shown in Figure 1.1.2), and Q is the volume rate of flow at which the oil is discharged from the tube. Equation (1.1.3) is derived, as we shall see later in this book [see Eq. (8.2.25)], using the principles of continuum mechanics.

PROBLEM 3 (HEAT TRANSFER)

We wish to determine the heat loss through the wall of a furnace. The wall typically consists of layers of brick, cement mortar, and cinder block (see Figure 1.1.3). Each of these materials provides varying degree of thermal resistance. The Fourier heat conduction law (see Section 8.3.1)

$$q = -k\frac{dT}{dx} \tag{1.1.4}$$

provides a relation between the heat flux q (heat flow per unit area) and gradient of temperature T. Here k denotes thermal conductivity ($1/k$ is the thermal resistance) of the material. The negative sign in Eq. (1.1.4) indicates that heat flows from

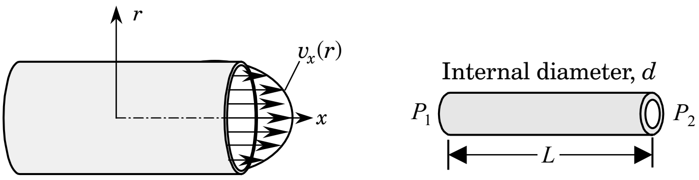

Figure 1.1.2. Measurement of viscosity of a fluid using capillary tube.

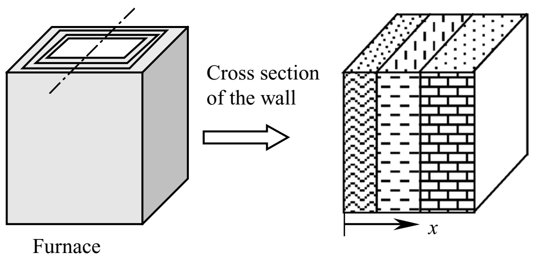

Figure 1.1.3. Heat transfer through a composite wall of a furnace.

high temperature region to low temperature region. Using the continuum mechanics equations, one can determine the heat loss when the temperatures inside and outside of the building are known. A building designer can select the materials as well as thicknesses of various components of the wall to reduce the heat loss (while ensuring necessary structural strength – a structural analysis aspect).

The previous examples provide some indication of the need for studying the mechanical response of materials under the influence of external loads. The response of a material is consistent with the laws of physics and the constitutive behavior of the material. This book has the objective of describing the physical principles and deriving the equations governing the stress and deformation of continuous materials and then solving some simple problems from various branches of engineering to illustrate the applications of the principles discussed and equations derived.

1.2 A Look Forward

The primary objective of this book is twofold: (1) use the physical principles to derive the equations that govern the motion and thermomechanical response of materials and (2) apply these equations for the solution of specific problems of linearized elasticity, heat transfer, and fluid mechanics. The governing equations for the study of deformation and stress of a continuous material are nothing but an analytical representation of the global laws of conservation of mass, momenta, and energy and the constitutive response of the continuum. They are applicable to all materials that are treated as a continuum. Tailoring these equations to particular problems and solving them constitutes the bulk of engineering analysis and design.

The study of motion and deformation of a continuum (or a “body” consisting of continuously distributed material) can be broadly classified into four basic categories:

(1) Kinematics (strain-displacement equations)
(2) Kinetics (conservation of momenta)
(3) Thermodynamics (first and second laws of thermodynamics)
(4) Constitutive equations (stress-strain relations)

Kinematics is a study of the geometric changes or deformation in a continuum, without the consideration of forces causing the deformation. *Kinetics* is the study of the static or dynamic equilibrium of forces and moments acting on a continuum,

Table 1.2.1. *The major four topics of study, physical principles and axioms used, resulting governing equations, and variables involved*

Topic of study	Physical principle	Resulting equations	Variables involved
1. Kinematics	None – based on geometric changes	Strain–displacement relations	Displacements and strains
		Strain rate–velocity relations	Velocities and strain rates
2. Kinetics	Conservation of linear momentum	Equations of motion	Stresses, velocities, and body forces
	Conservation of angular momentum	Symmetry of stress tensor	Stresses
3. Thermodynamics	First law	Energy equation	Temperature, heat flux, stresses, heat generation, and velocities
	Second law	Clausius–Duhem inequality	Temperature, heat flux, and entropy
4. Constitutive equations (not all relations are listed)	Constitutive axioms	Hooke's law	Stresses, strains, heat flux and temperature
		Newtonian fluids	Stresses, pressure, velocities
		Fourier's law	Heat flux and temperature
		Equations of state	Density, pressure, temperature

using the principles of conservation of momenta. This study leads to equations of motion as well as the symmetry of stress tensor in the absence of body couples. *Thermodynamic principles* are concerned with the conservation of energy and relations among heat, mechanical work, and thermodynamic properties of the continuum. *Constitutive equations* describe thermomechanical behavior of the material of the continuum, and they relate the dependent variables introduced in the kinetic description to those introduced in the kinematic and thermodynamic descriptions. Table 1.2.1 provides a brief summary of the relationship between physical principles and governing equations, and physical entities involved in the equations.

1.3 Summary

In this chapter, the concept of a continuous medium is discussed, and the major objectives of the present study, namely, to use the physical principles to derive the equations governing a continuous medium and to present application of the equations in the solution of specific problems of linearized elasticity, heat transfer, and fluid mechanics, are presented. The study of physical principles is broadly divided into four topics, as outlined in Table 1.2.1. These four topics form the subject of Chapters 3 through 6, respectively. Mathematical formulation of the governing equations of a continuous medium necessarily requires the use of vectors and tensors, objects that facilitate invariant analytical formulation of the natural laws. Therefore, it is useful to study certain operational properties of vectors and tensors first. Chapter 2 is dedicated for this purpose.

While the present book is self-contained for an introduction to continuum mechanics, there are other books that may provide an advanced treatment of the subject. Interested readers may consult the titles listed in the reference list at the end of the book.

PROBLEMS

1.1 Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a}, \tag{1}$$

where $\mathbf{F}$ is the net force acting on the body, m mass of the body, and $\mathbf{a}$ the acceleration of the body in the direction of the net force. Use Eq. (1) to determine the governing equation of a free-falling body. Consider only the forces due to gravity and the air resistance, which is assumed to be linearly proportional to the velocity of the falling body.

1.2 Consider steady-state heat transfer through a cylindrical bar of nonuniform cross section. The bar is subject to a known temperature T_0 (°C) at the left end and exposed, both on the surface and at the right end, to a medium (such as cooling fluid or air) at temperature T_∞. Assume that temperature is uniform at any section of the bar, $T = T(x)$. Use the principle of conservation of energy (which requires that the rate of change (increase) of internal energy is equal to the sum of heat gained by conduction, convection, and internal heat generation) to a typical element of the bar (see Figure P1.2) to derive the governing equations of the problem.

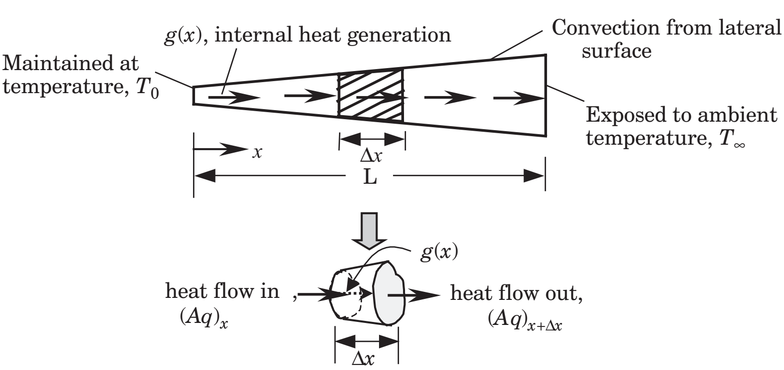

Figure P1.2.

1.3 The Euler–Bernoulli hypothesis concerning the kinematics of bending deformation of a beam assumes that straight lines perpendicular to the beam axis before deformation remain (1) straight, (2) perpendicular to the tangent line to the beam axis, and (3) inextensible during deformation. These assumptions lead to the following displacement field:

$$u_1 = -z\frac{dw}{dx}, \quad u_2 = 0, \quad u_3 = w(x), \tag{1}$$

where (u_1, u_2, u_3) are the displacements of a point (x, y, z) along the x, y, and z coordinates, respectively, and w is the vertical displacement of the beam at point $(x, 0, 0)$. Suppose that the beam is subjected to distributed transverse load $q(x)$. Determine the governing equation by summing the forces and moments on an element of the beam (see Figure P1.3). Note that the sign convention for the moment and shear force are based on the definitions

$$V = \int_A \sigma_{xz}\, dA, \quad M = \int_A z\,\sigma_{xx}\, dA,$$

and it may not agree with the sign convention used in some mechanics of materials books.

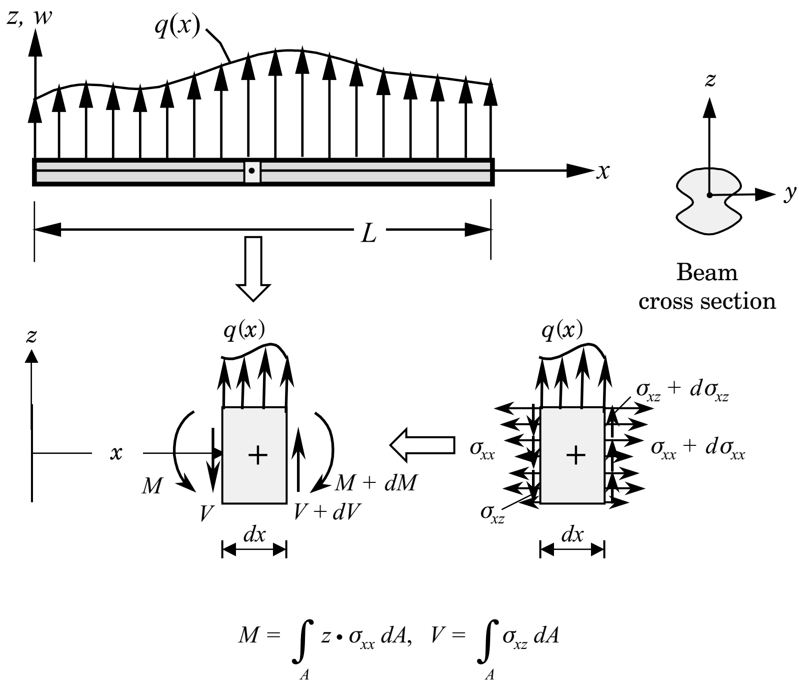

Figure P1.3.

1.4 A cylindrical storage tank of diameter D contains a liquid column of height $h(x, t)$. Liquid is supplied to the tank at a rate of q_i (m^3/day) and drained at a rate of q_0 (m^3/day). Use the principle of conservation of mass to obtain the equation governing the flow problem.

2 Vectors and Tensors

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert

2.1 Background and Overview

In the mathematical description of equations governing a continuous medium, we derive relations between various quantities that characterize the stress and deformation of the continuum by means of the laws of nature (such as Newton's laws, conservation of energy, and so on). As a means of expressing a natural law, a coordinate system in a chosen frame of reference is often introduced. The mathematical form of the law thus depends on the chosen coordinate system and may appear different in another type of coordinate system. The laws of nature, however, should be independent of the choice of a coordinate system, and we may seek to represent the law in a manner independent of a particular coordinate system. A way of doing this is provided by vector and tensor analysis. When vector notation is used, a particular coordinate system need not be introduced. Consequently, the use of vector notation in formulating natural laws leaves them *invariant* to coordinate transformations. A study of physical phenomena by means of vector equations often leads to a deeper understanding of the problem in addition to bringing simplicity and versatility into the analysis.

In basic engineering courses, the term *vector* is used often to imply a *physical* vector that has 'magnitude and direction and satisfy the parallelogram law of addition.' In mathematics, vectors are more abstract objects than physical vectors. Like physical vectors, *tensors* are more general objects that are endowed with a magnitude and multiple direction(s) and satisfy rules of tensor addition and scalar multiplication. In fact, physical vectors are often termed the *first-order tensors*. As will be shown shortly, the specification of a stress component (i.e., force per unit area) requires a magnitude and two directions – one normal to the plane on which the stress component is measured and the other is its direction – to specify it uniquely.

This chapter is dedicated to a review of algebra and calculus of physical vectors and tensors. Those who are familiar with the material covered in any of the sections may skip them and go to the next section or Chapter 3.

2.2 Vector Algebra

In this section, we present a review of the formal definition of a geometric (or physical) vector, discuss various products of vectors and physically interpret them, introduce index notation to simplify representations of vectors in terms of their components as well as vector operations, and develop transformation equations among the components of a vector expressed in two different coordinate systems. Many of these concepts, with the exception of the index notation, may be familiar to most students of engineering, physics, and mathematics and may be skipped.

2.2.1 Definition of a Vector

The quantities encountered in analytical description of physical phenomena may be classified into two groups according to the information needed to specify them completely: scalars and nonscalars. The scalars are given by a single number. Nonscalars have not only a magnitude specified but also additional information, such as direction. Nonscalars that obey certain rules (such as the parallelogram law of addition) are called *vectors*. Not all nonscalar quantities are vectors (e.g., a finite rotation is not a vector).

A physical vector is often shown as a directed line segment with an arrow head at the end of the line. The length of the line represents the magnitude of the vector and the arrow indicates the direction. In written or typed material, it is customary to place an arrow over the letter denoting the vector, such as $\vec{A}$. In printed material, the vector letter is commonly denoted by a boldface letter $\mathbf{A}$, such as used in this book. The magnitude of the vector $\mathbf{A}$ is denoted by $|\mathbf{A}|$, $\|\mathbf{A}\|$, or A. Magnitude of a vector is a scalar.

A vector of unit length is called a *unit vector*. The unit vector along $\mathbf{A}$ may be defined as follows:

$$\hat{\mathbf{e}}_A = \frac{\mathbf{A}}{A}. \tag{2.2.1}$$

We may now write

$$\mathbf{A} = A\,\hat{\mathbf{e}}_A. \tag{2.2.2}$$

Thus *any vector may be represented as a product of its magnitude and a unit vector* along the vector. A unit vector is used to designate direction. It does not have any physical dimensions. We denote a unit vector by a "hat" (caret) above the boldface letter, $\hat{\mathbf{e}}$. A vector of zero magnitude is called a *zero vector* or a *null vector*. All null vectors are considered equal to each other without consideration as to direction. Note that a light face zero, 0, is a scalar and boldface zero, $\mathbf{0}$, is the zero vector.

2.2.1.1 Vector Addition

Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be any vectors. Then there exists a vector $\mathbf{A}+\mathbf{B}$, called sum of $\mathbf{A}$ and $\mathbf{B}$, such that

$$
\begin{aligned}
&\textbf{(1)}\ \ \mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}\ \ \text{(commutative)}.\\
&\textbf{(2)}\ \ (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})\ \ \text{(associative)}.\\
&\textbf{(3)}\ \ \text{there exists a unique vector, } \mathbf{0}\text{, independent of } \mathbf{A} \text{ such that}\\
&\qquad \mathbf{A}+\mathbf{0}=\mathbf{A}\ \ \text{(existence of zero vector)}.\\
&\textbf{(4)}\ \ \text{to every vector } \mathbf{A} \text{ there exists a unique vector } -\mathbf{A}\\
&\qquad \text{(that depends on } \mathbf{A}\text{) such that}\\
&\qquad \mathbf{A}+(-\mathbf{A})=\mathbf{0}\ \ \text{(existence of negative vector)}.
\end{aligned}
\tag{2.2.3}
$$

The negative vector $-\mathbf{A}$ has the same magnitude as $\mathbf{A}$ but has the opposite *sense*. Subtraction of vectors is carried out along the same lines. To form the difference $\mathbf{A}-\mathbf{B}$, we write $\mathbf{A}+(-\mathbf{B})$ and subtraction reduces to the operation of addition.

2.2.1.2 Multiplication of Vector by Scalar

Let $\mathbf{A}$ and $\mathbf{B}$ be vectors and α and β be real numbers (scalars). To every vector $\mathbf{A}$ and every real number α, there corresponds a unique vector $\alpha\mathbf{A}$ such that

$$
\begin{aligned}
&\textbf{(1)}\ \ \alpha(\beta\mathbf{A})=(\alpha\beta)\mathbf{A}\ \text{(associative)}.\\
&\textbf{(2)}\ \ (\alpha+\beta)\mathbf{A}=\alpha\mathbf{A}+\beta\mathbf{A}\ \text{(distributive scalar addition)}.\\
&\textbf{(3)}\ \ \alpha(\mathbf{A}+\mathbf{B})=\alpha\mathbf{A}+\alpha\mathbf{B}\ \text{(distributive vector addition)}.\\
&\textbf{(4)}\ \ 1\cdot\mathbf{A}=\mathbf{A}\cdot 1=\mathbf{A},\ \ 0\cdot\mathbf{A}=\mathbf{0}.
\end{aligned}
\tag{2.2.4}
$$

Equations (2.2.3) and (2.2.4) clearly show that the laws that govern addition, subtraction, and scalar multiplication of vectors are identical with those governing the operations of scalar algebra.

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if their magnitudes are equal, $|\mathbf{A}|=|\mathbf{B}|$, *and* if their directions are equal. Consequently, a vector is not changed if it is moved parallel to itself. This means that the position of a vector in space, that is, the point from which the line segment is drawn (or the end without arrowhead), may be chosen arbitrarily. In certain applications, however, the actual point of location of a vector may be important, for instance, a moment or a force acting on a body. A vector associated with a given point is known as a *localized* or *bound vector*. A finite rotation of a rigid body is not a vector although infinitesimal rotations are. That vectors can be represented graphically is an *incidental* rather than a fundamental feature of the vector concept.

2.2.1.3 Linear Independence of Vectors

The concepts of collinear and coplanar vectors can be stated in algebraic terms. A set of n vectors is said to be *linearly dependent* if a set of n numbers $\beta_1, \beta_2, \ldots, \beta_n$ can be found such that

$$\beta_1\mathbf{A}_1+\beta_2\mathbf{A}_2+\cdots+\beta_n\mathbf{A}_n=\mathbf{0}, \tag{2.2.5}$$