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Encyclopedia of Mathematics and its Applications

Aperiodic Order

Volume 1: A Mathematical Invitation

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Foreword by Roger Penrose

In a famous address to the 1900 International Congress of Mathematicians, held in Paris, the great mathematician David Hilbert announced a list of 23 unsolved mathematical problems, many of which shaped the subsequent course of mathematics for the 20$^{\text{th}}$ century. It would have been clear at the time that several of the problems concerned issues of profound mainstream mathematical interest. Some of the others may have seemed, then, more like curious mathematical side-issues; yet Hilbert showed a remarkable sensitivity in realising that within such problems were matters of genuine potential, mathematical subtlety and importance.

In this latter category was Problem 18, which raises issues of the filling of space with congruent shapes. Among other matters (such as the *Kepler conjecture* concerning the close-packing of spheres[1]) was the question of whether there exists a polyhedron which tiles Euclidean 3-space, but only in a way that it is not the fundamental domain of any space group — that is to say, must every tiling by that polyhedron be necessarily isohedral, which would mean that every instance of the polyhedron is obtainable from every other, through a Euclidean motion of the entire tiling pattern into itself (i.e., all polyhedra in the tiling would thereby be on an 'equal footing' with respect to the pattern as a whole). Such shapes which tile space, but only in ways that are *not* isohedral, are now known as *anisohedal* prototiles (where the word 'prototile' simply means a *tile shape*, in current terminology). In asking this question in three (and more) dimensions, Hilbert was probably assuming — incorrectly, as it turned out — that no anisohedral prototile could exist in two dimensions. No doubt it was felt at that time that little could be mathematically interesting or difficult concerning tiling with plane shapes. How greatly our views on this matter have now changed!

The first 3-dimensional anisohedral tile was found by Karl Reinhardt in 1928, and then in 1935 Heinrich Heesch found a 2-dimensions example. My own first serious interest into the area of plane tilings was exhibited in an article I wrote with my father, in 1958, which presented a small collection of

[1]Solved, in 1998, by a computer-aided proof by Thomas Hales, following an approach suggested by László Fejes Tóth in 1953.

somewhat unusual 'Puzzles for Christmas' [PP58b]. Puzzle 6 asked for the (unique) tiling patterns provided by each of seven different prototiles, the last five of these being anisohedral — although I was not at that time aware either of the terminology or of the inherent interest in such features.

I had played with such things, on and off, for several years before that, and it had been in 1954 that I had first come across the graphic work of Maurits C. Escher, with his many representations of tiling patterns using birds, fish and many other curious and fascinating designs (all isohedral, in fact). This was while attending the 12th International Congress of Mathematicians held in Amsterdam (as a second-year graduate student at Cambridge). I was amazed by much that I saw there, and as a result, I decided to try my hand at creating 'Escher-like' pictures of one kind or another, having written another article with my father on one aspect of these [PP58a]. We cited Escher's work as depicted in the Amsterdam exhibition and, as a result of this, my father entered into a correspondence with Escher.

In around 1962, I happened to be driving through the Netherlands, and I telephoned Escher on the off-chance that I might be able to visit him. He most obligingly invited me (and my then wife Joan) to tea, showing me a great number of his amazing creations, one of which I was offered as a gift. In exchange, I left a number of identically-shaped wooden pieces with him, set as a puzzle for him to see how to use these prototiles to tile the entire Euclidean plane. The task was not an immediately obvious one, the shape requiring 12 different orientations (six being inverted) before repeating, the tiling being necessarily non-isohedral. Escher used a modified version of this tiling pattern in his final (and only non-isohedral) print.[2]

The exploration of anisohedral prototiles has moved on enormously since the early work of Reinhardt and Heesch. All the anisohedral prototiles that I have referred to so far are what are called *2-isohedral*, where a k-isohedral prototile would be one which can tile the entire space in a way that the pattern involves k transitivity classes, but not in a way where the pattern involves fewer than k transitivity classes (so that the tile necessarily takes up k distinct relationships to the pattern as a whole). In 2003, Joseph Myers exhibited a remarkable 10-isohedral tile — apparently the current record.[3]

Subsequent to my early concern with such things as anisohedral prototiles, I became interested in the possibility of prototiles that tile the plane in hierarchical ways (such as those that later became known as *rep-tiles*), and this route led me to the aperiodic sets of prototiles now referred to as 'Penrose tiles' where the term *aperiodic* refers to the fact that whereas they do tile

[2]'Ghosts'; see, for example, top left of [CEPT86, p. 395], and my own article 'Escher and the visual representation of mathematical ideas' in the same volume.

[3]See the excellent article [Goo11] by Chaim Goodman-Strauss.

the entire plane, they do so only without translational symmetry (so that, in a sense, they constitute an '∞-isohedral' pair). My own approach to these issues was very direct and geometrical, and not the result of any deep mathematical underpinning, such as was later provided by Nicolaas de Bruijn's [dBr81] very fruitful and revealing subsequent procedure for obtaining such tiling patterns by slicing and projecting higher-dimensional cubic lattices. Although there was no such sophisticated mathematical input into the initial discovery of these particular prototiles, my vague awareness of the profound logical considerations of Hao Wang, as developed by Robert Berger, and leading to Raphael Robinson's aperiodic 6-prototile set [Rob71] could well have influenced my anticipation that the original non-periodic pentagon pattern that I had found [Pen74, Fig. 4] could be *forced* as the tiling arrangement of an aperiodic 6-prototile set, and subsequently that this set could be reduced to an aperiodic 2-prototile set.

Of more direct (although largely unconscious) influence was my much earlier acquaintance with Johannes Kepler's very remarkable 1619 picture exhibiting various non-crystallographic tiling patterns [Kep19]. I did not have these in mind when I first found my own aperiodic sets early in 1974, but I had seen Kepler's designs many years before, and I believe that they strongly influenced my attitude to the fruitfulness of pentagonal tilings. It was only some years after 1974 that I realised the extraordinarily close relationship between Kepler's configuration 'Aa' and my own pentagonal tiling, where the configuration of line segments constituting Kepler's entire configuration, *without any exceptions*, can be found within those of my own pentagonal tiling. I have always been intrigued as to how Kepler intended his pattern 'Aa' to be continued, and I believe it is quite within plausible possibility that he had in mind some sort of hierarchical continuation similar to that of my own pattern.

Kepler was clearly interested in the different kinds of symmetry that could co-exist with the packing of shapes together in a systematic way, possibly in relation to some kind of atomic underpinning that might be relevant to biological as well as crystalline structures. The non-crystalline symmetry underlying my own tilings is indeed intimately related to their lack of periodicity, a feature distinguishing them from the earlier aperiodic sets of Berger and Robinson. Many Islamic patterns also contain elements with the non-crystallographic 5-fold and 10-fold symmetry, as well as 8-fold and 12-fold, and I had found these fascinating, but I was not aware of any such patterns containing extended regions with such symmetry, and I do not think that these things influenced me significantly in my own quasi-symmetric designs.

I suspect that Kepler's influence on me might have been similar to that which my own tiling patterns could have had on Dan Shechtman. He once told me that when he first came across his puzzling 10-fold-symmetric diffraction patterns (those that led him to his revolutionary insights that won him the 2011 Nobel Prize in Chemistry), he did not have my tiling patterns in mind. Yet, he said he had been aware of them, and I like to think that they may have unconsciously influenced him to be favourably disposed towards the presence of a genuinely novel type of 'crystallographic' atomic arrangement with an underlying 10-fold or 5-fold symmetry, as turned out to be the case.

My own tilings, and those found by Robert Ammann very shortly afterwards, were fortunately discovered in time to be incorporated and beautifully described by Branko Grünbaum and Geoffrey Shephard in their classic and, at that time, comprehensive 1987 work *Tilings and Patterns*. Before this, there had been little organisation in the subject or in its terminology. The current work, by Michael Baake and Uwe Grimm, provides a most worthy continuation of Grünbaum and Shephard's achievement, providing, as it does, an excellent overview of the subject of aperiodic order that has grown up since that time. The mathematical underpinnings of the aperiodic order underlying the non-crystallographic patterns exhibited by these early aperiodic tile sets has been vastly extended. In addition to the 5-fold (or 10-fold) and Ammann (and Beenker) 8-fold quasi-symmetric patterns, which arrived in time for inclusion in *Tilings and Patterns*, we now have, illustrated here, several 12-fold examples found by Schlottmann, Gähler, Socolar and others.

Moreover, we find here a study not only of these quasi-symmetries, which appear to underlie the actual atomic arrangements of quasi-crystals, but also of many other types of quasi-symmetric patterns. For example, there is the 7-fold case, as exhibited by the 3-prototile set constructed by Ludwig Danzer, which is aperiodic if we restrict the tilings to conform to a certain atlas of vertex stars, or else if we require that we construct the tiling by following specific *inflation* rules. Most of the early aperiodic tilings (including the rep-tiles, the Robinson 6-prototile set, Amman's examples, and my own), were originally based upon inflation rules where, in effect, the prototiles can be collected into larger versions of themselves, and the entire pattern is built up in this hierarchical way. More recent tilings that can be obtained by inflation are also described here, such as the remarkable hexagonal prototile of Joan Taylor and a somewhat similar earlier one of my own — both being aperiodic via second-order matching rules. The inflation scaling factor is referred to as the *inflation number*, and for the 5-, 8-, 10-, and 12-fold quasi-symmetries, this inflation number is an example of what is referred to as a *Pisot* (or Pisot–Vijayaraghavan) number (a real algebraic integer larger than 1 whose conjugates all lie within the unit circle in the complex plane). More unsettled-looking tilings, such as Danzer's, arise with *non*-Pisot inflation numbers.

This illustrates a connection between tilings and *number theory* and *complex analysis*, and many connections with other areas of mathematics are also provided here, such as that arising from the work of Nicolaas de Bruijn referred to above. Intriguing relationships to various other areas of mathematics are also demonstrated here. Group theory has an obvious importance, and Peter Kramer's group theoretic approach to the projection method was particularly noteworthy in relation to 3-dimensional quasicrystals. But we also find roles for Fourier analysis, a subject which is highly relevant to the subject of crystallography, where diffraction patterns play key roles. Thus, we find here many windows into powerful areas of mathematics which are often as unexpected as they are fascinating. Most importantly, there is the additional attraction that so much of it can be illustrated in actual visual images that are beautiful to behold and for which much of their mathematical quality can be discerned by simply looking at them. We have surely moved a long way from the simple question of whether an anisohedral prototile can exist! One can but wonder what David Hilbert would have made of it all.

Roger Penrose

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Michael Baake and Uwe Grimm
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Preface

The theory of aperiodic order is still a relatively young field of mathematics. It has grown rapidly over the past three decades following the experimental discovery of aperiodically ordered materials. This is the introductory volume of a book series that attempts to provide a comprehensive account of the field, which is still developing. We entitled this volume 'a mathematical invitation' because we hope that it will inspire readers to enter the fascinating (and largely still to be explored) world of aperiodic order and provide the background that enables them to follow the current developments. Subsequent volumes will address particular as well as complementary topics in more depth, in the form of selected survey articles contributed by expert authors. While the scope and the details of the later volumes are still evolving, the second volume will focus on crystallography and almost periodicity, and the third is planned to expand on model sets and dynamical systems.

It was our aim to keep this introductory volume at a relatively elementary level, and to make the subject accessible to both mathematicians and physicists, which requires a certain compromise for the exposition. Consequently, we ask readers with a mathematical background for their patience for our largely constructive and often example-motivated approach, which sometimes also requires substantial explicit calculations. In particular, we do not strive at a formal exposition at maximal generality, but rather at emphasising the meaning of the formalism at each step. Similarly, people with a background in the physical sciences are kindly asked to bear with us on our journey through aperiodic order, which is intentionally more rigorous than the standard physics literature. While we aim at a certain level of completeness (and at substantiating some of the 'folklore'), we will not spell out all proofs, but sometimes refer to the original literature.

For a number of years, there have been parallel developments in mathematics and physics, which took limited notice of each other. It is one of our aims to bring these developments together, which in particular means a more physics oriented selection of topics with a stronger mathematically oriented exposition. Within physics, we were influenced mostly by the work of Peter Kramer, who pioneered the symmetry-oriented construction of non-periodic tilings by the projection method. Within mathematics, we would like to highlight the early contributions by the late Peter Pleasants, in particular his number-theoretic approach to the projection method. With hindsight, it is the most natural continuation of the pioneering (but only later appreciated) work of Yves Meyer, and has influenced our way of thinking and our presentation significantly. Roger Penrose's famous fivefold tiling provided a paradigm that not only shows the aesthetic appeal of aperiodically ordered structures, but has become an influential guiding example for many later and current investigations, also with regard to applications. We also build on the geometric insight of Robert Ammann and Ludwig Danzer, who constructed many important tilings in two and three dimensions. More recently, the advance of the field, in particular in its mathematical flavour, was perhaps most stimulated by the contributions of Jeffrey Lagarias and Robert Moody, which will also be reflected by our exposition.

A number of people read and commented on drafts of various parts of this book, which resulted in numerous corrections and improvements. In particular, we would like to thank Shelomo Ben-Abraham, Paolo Bugarin, David Damanik, Aernout van Enter, Dirk Frettlöh, Franz Gähler, Svenja Glied, Manuela Heuer, Christian Huck, Tobias Jakobi, Daniel Lenz, Reinhard Lück, Claudia Lütkehölter, Markus Moll, Robert Moody, Natascha Neumärker, Johan Nilsson, Arthur (Robbie) Robinson, Johannes Roth, Boris Solomyak, Joan Taylor, Venta Terauds and Peter Zeiner for their valuable input. Special thanks go to Dirk Frettlöh for contributing to the parts on Danzer's icosahedral ABCK tiling, to Franz Gähler for his help with the literature on local rules, and to Egon Schulte for his encouragement during the writing process. All remaining errors in the manuscript are ours.

Finally, it is our pleasure to thank the Department of Mathematics and Physics at the University of Tasmania, Hobart, for its hospitality and for providing a stimulating working environment to get this project off the ground. During the period of writing, we received support from various sources, including the German Research Foundation (DFG), the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna, the Engineering and Physical Sciences Research Council (EPSRC), the Leverhulme Trust and the Royal Society.

The revised second printing gave us the opportunity to include a number of minor corrections and additions. We are grateful to Robert Moody and Nicolae Strungaru for their suggestions. Moreover, a list of definitions has been added, and the references have been updated.

Michael Baake and Uwe Grimm