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Viatcheslav Mukhanov and Sergei Winitzki
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Part I

Canonical quantization and particle production

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Overview: a taste of quantum fields

Summary Quantum fields as a set of harmonic oscillators. Vacuum state. Particle interpretation of field theory. Examples of particle production by external fields.

We begin with a few elementary observations concerning the vacuum in quantum field theory.

1.1 Classical field

A classical field is described by a function $\phi(\mathbf{x}, t)$, where $\mathbf{x}$ is a three-dimensional coordinate in space and t is the time. At every point the function $\phi(\mathbf{x}, t)$ takes values in some finite-dimensional "configuration space" and can be a scalar, vector, or tensor.

The simplest example is a real scalar field $\phi(\mathbf{x}, t)$ whose strength is characterized by real numbers. A free massive scalar field satisfies the Klein–Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2 \phi}{\partial x_j^2} + m^2 \phi \equiv \ddot{\phi} - \Delta \phi + m^2 \phi = 0, \qquad (1.1)$$

which has a unique solution $\phi(\mathbf{x}, t)$ for $t > t_0$ provided that the initial conditions $\phi(\mathbf{x}, t_0)$ and $\dot{\phi}(\mathbf{x}, t_0)$ are specified.

Formally one can describe a free scalar field as a set of decoupled "harmonic oscillators." To explain why this is so it is convenient to begin by considering a field $\phi(\mathbf{x}, t)$ not in infinite space but in a box of finite volume V, with some boundary conditions imposed on the field ϕ. The volume V should be large enough to avoid artifacts induced by the finite size of the box or by physically irrelevant boundary conditions. For example, one might choose the box as a cube

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with sides of length L and volume $V = L^3$, and impose the *periodic* boundary conditions,

$$\phi\,(x=0,y,z,t) = \phi\,(x=L,y,z,t)$$

and similarly for y and z. The Fourier decomposition is then

$$\phi\,(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \phi_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1.2}$$

where the sum goes over three-dimensional wavenumbers $\mathbf{k}$ with components

$$k_x = \frac{2\pi n_x}{L}, \quad n_x = 0, \pm 1, \pm 2, \ldots$$

and similarly for k_y and k_z. The normalization factor $\sqrt{V}$ in equation (1.2) is chosen to simplify formulae (in principle, one could rescale the modes $\phi_{\mathbf{k}}$ by any constant). Substituting (1.2) into equation (1.1), we find that this equation is replaced by an infinite set of decoupled ordinary differential equations:

$$\ddot{\phi}_{\mathbf{k}} + \left(k^2 + m^2\right) \phi_{\mathbf{k}} = 0,$$

with one equation for each $\mathbf{k}$. In other words, each complex function $\phi_{\mathbf{k}}(t)$ satisfies the harmonic oscillator equation with the frequency

$$\omega_k \equiv \sqrt{k^2 + m^2},$$

where $k \equiv |\mathbf{k}|$. The "oscillators" with complex coordinates $\phi_{\mathbf{k}}$ "move" not in real three-dimensional space but in the *configuration space* and characterize the strength of the field ϕ. The total energy of the field ϕ in the box is simply equal to the sum of energies of all oscillators $\phi_{\mathbf{k}}$,

$$E = \sum_{\mathbf{k}} \left[\frac{1}{2} \left| \dot{\phi}_{\mathbf{k}} \right|^2 + \frac{1}{2} \omega_k^2 \left| \phi_{\mathbf{k}} \right|^2 \right].$$

In the limit of infinite space when $V \to \infty$ the sum in (1.2) is replaced by the integral over all wavenumbers $\mathbf{k}$,

$$\phi\,(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{\mathbf{k}}(t). \tag{1.3}$$

1.2 Quantum field and its vacuum state

The quantization of a free scalar field is mathematically equivalent to quantizing an infinite set of decoupled harmonic oscillators.

Harmonic oscillator A classical harmonic oscillator is described by a coordinate $q(t)$ satisfying

$$\ddot{q} + \omega^2 q = 0. \tag{1.4}$$

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The solution of this equation is unique if we specify initial conditions $q(t_0)$ and $\dot{q}(t_0)$. We may identify the "ground state" of an oscillator as the state without motion, i.e. $q(t) \equiv 0$. This lowest-energy state is the solution of the classical equation (1.4) with the initial conditions $q(0) = \dot{q}(0) = 0$.

When the oscillator is quantized, the classical coordinate q and the momentum $p = \dot{q}$ (for simplicity, we assume that the oscillator has a unit mass) are replaced by operators $\hat{q}(t)$ and $\hat{p}(t)$ satisfying the Heisenberg commutation relation

$$[\hat{q}(t), \hat{p}(t)] = [\hat{q}(t), \dot{\hat{q}}(t)] = i\hbar. \tag{1.5}$$

The solution $\hat{q}(t) \equiv 0$ does not satisfy the commutation relation. In fact, the oscillator's coordinate always fluctuates. The ground state with the lowest energy is described by the normalized wave function

$$\psi(q) = \left[\frac{\omega}{\pi\hbar}\right]^{\frac{1}{4}} \exp\left(-\frac{\omega q^2}{2\hbar}\right).$$

The energy of this minimal excitation state, called the *zero-point energy*, is $E_0 = \frac{1}{2}\hbar\omega$. The typical amplitude of fluctuations in the ground state is $\delta q \sim \sqrt{\hbar/\omega}$ and the measured trajectories $q(t)$ resemble a random walk around $q = 0$.

Field quantization In the case of a field, each mode $\phi_{\mathbf{k}}(t)$ is quantized as a separate harmonic oscillator. The classical "coordinates" $\phi_{\mathbf{k}}$ and the corresponding conjugated momenta $\pi_{\mathbf{k}} \equiv \dot{\phi}^*_{\mathbf{k}}$ are replaced by operators $\hat{\phi}_{\mathbf{k}}$, $\hat{\pi}_{\mathbf{k}}$. In a finite box they satisfy the following equal-time commutation relations:

$$\left[\hat{\phi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{k}'}(t)\right] = i\delta_{\mathbf{k},-\mathbf{k}'},$$

where $\delta_{\mathbf{k},-\mathbf{k}'}$ is the Kronecker symbol equal to unity when $\mathbf{k} = -\mathbf{k}'$ and zero otherwise. In the limit of infinite volume the commutation relations become

$$\left[\hat{\phi}_{\mathbf{k}}(t), \hat{\pi}_{\mathbf{k}'}(t)\right] = i\delta\left(\mathbf{k} + \mathbf{k}'\right), \tag{1.6}$$

where $\delta(\mathbf{k} + \mathbf{k}')$ is the Dirac δ function. To simplify the formulae, we shall almost always use the units in which $\hbar = c = 1$.

Vacuum state The *vacuum* is a state corresponding to the intuitive notions of "the absence of anything" or "an empty space." Generally, the vacuum is defined as the state with the lowest possible energy. In the case of a classical field the vacuum is a state where the field is absent, that is, $\phi(\mathbf{x}, t) = 0$. This is a solution of the classical equations of motion. When the field is quantized it becomes impossible to satisfy simultaneously the equations of motion for the operator $\hat{\phi}$ and the commutation relations by $\hat{\phi}(\mathbf{x}, t) = 0$. Therefore, the field always fluctuates and has a nonvanishing value even in a state with the minimal possible energy.

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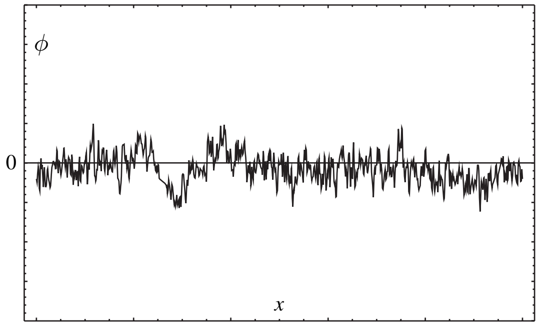

Fig. 1.1 A field configuration $\phi(x)$ that could be measured in the vacuum state.

Since all modes $\phi_{\mathbf{k}}$ are decoupled, the ground state of the field can be characterized by a *wave functional* which is the product of an infinite number of wave functions, each describing the ground state of a harmonic oscillator with the corresponding wavenumber $\mathbf{k}$:

$$\Psi[\phi] \propto \prod_{\mathbf{k}} \exp\left(-\frac{\omega_k |\phi_{\mathbf{k}}|^2}{2}\right) = \exp\left[-\frac{1}{2}\sum_{\mathbf{k}} \omega_k |\phi_{\mathbf{k}}|^2\right]. \tag{1.7}$$

The ground state of the field has the minimum energy and is called the vacuum state. Strictly speaking, equation (1.7) is valid only for a field quantized in a box. Note that if we had normalized the Fourier components $\phi_{\mathbf{k}}$ in equation (1.2) differently, then there would be a volume factor in front of ω_k.

The square of the wave function (1.7) gives us the probability density for measuring a certain field configuration $\phi(\mathbf{x})$. This probability is independent of time t. The field fluctuates in the vacuum state and the field configurations can be visualized as small random deviations from zero (see Fig. 1.1).

When the volume of the box becomes very large, we have to replace sums by integrals,

$$\sum_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d^3\mathbf{k}, \quad \phi_{\mathbf{k}} \to \sqrt{\frac{(2\pi)^3}{V}}\phi_{\mathbf{k}}, \tag{1.8}$$

and the wave functional (1.7) becomes

$$\Psi[\phi] \propto \exp\left[-\frac{1}{2}\int d^3\mathbf{k}\, |\phi_{\mathbf{k}}|^2\, \omega_k\right]. \tag{1.9}$$

Exercise 1.1
The vacuum wave functional (1.9) contains the integral

$$I \equiv \int d^3\mathbf{k}\, |\phi_{\mathbf{k}}|^2 \sqrt{k^2+m^2}, \tag{1.10}$$

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where $\phi_{\mathbf{k}}$ are defined in equation (1.3). This integral can be expressed directly in terms of the function $\phi(\mathbf{x})$,

$$I = \int d^3\mathbf{x}\, d^3\mathbf{y}\, \phi(\mathbf{x})\, K(\mathbf{x}, \mathbf{y})\, \phi(\mathbf{y}).$$

Determine the required kernel $K(\mathbf{x}, \mathbf{y})$.

1.3 The vacuum energy

Let us compute the energy of a free scalar field in the vacuum state. Each oscillator $\hat{\phi}_{\mathbf{k}}$ is in its ground state and has energy $\frac{1}{2}\omega_k$, so that the total zero-point energy of the field in a box of finite volume V is

$$E_0 = \sum_{\mathbf{k}} \frac{1}{2}\omega_k.$$

Taking the limit $V \to \infty$ and replacing the sum by an integral according to (1.8), we obtain the following expression for the vacuum energy density,

$$\frac{E_0}{V} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2}\omega_k. \tag{1.11}$$

The integral diverges at the upper bound as k^4. Taken at face value, this would indicate an infinite vacuum energy *density*. If we impose an ultraviolet cutoff, for example, at the Planckian scale, where one expects quantum gravity to induce new physics, then the vacuum energy density is of order unity in the Planck units. This corresponds to a mass density of about $10^{94}\,\mathrm{g/cm^3}$. We recall that the mass of the entire observable Universe is only $\sim 10^{55}\mathrm{g}$! Therefore, if the vacuum energy contributes to the gravitational field, such a huge energy density is in obvious contradiction with observations.

The standard way to resolve this problem is to *postulate* that the vacuum energy density given in (1.11) does not contribute to the gravity. Another way to avoid this problem is to consider a supersymmetric theory where every field has a supersymmetric partner that contributes an equal amount to the vacuum energy with an opposite sign. However, experiments show that supersymmetry must be broken at some energy scale that is larger than the energy currently accessible to particle accelerators. This leads to a mismatch of the superpartner contributions to the vacuum energy of order the supersymmetry breaking scale, which is still too large when compared with observational limits. Therefore the supersymmetric solution of the vacuum energy problem is not immediately successful.

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1.4 Quantum vacuum fluctuations

Amplitude of fluctuations As we found above, the typical amplitude of quantum fluctuations for the mode **k** is

$$\delta\phi_{\mathbf{k}} \equiv \sqrt{\left\langle |\phi_{\mathbf{k}}|^2 \right\rangle} \sim \omega_k^{-1/2}. \tag{1.12}$$

Field values cannot be measured at a point; in a realistic experiment, only their values, averaged over a finite region of space, are measured. Let us consider the average value of a field $\phi(\mathbf{x})$ in a cube-shaped region of volume L^3,

$$\phi_L \equiv \frac{1}{L^3} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dz\, \phi(\mathbf{x}) .$$

Exercise 1.2

Justify the following order-of-magnitude estimate of the typical amplitude of fluctuations $\delta\phi_L$,

$$\delta\phi_L \sim \left[(\delta\phi_{\mathbf{k}})^2 k^3 \right]^{1/2}, \quad k = L^{-1},$$

where $k \equiv |\mathbf{k}|$ and $\delta\phi_{\mathbf{k}}$ is the typical amplitude of fluctuations in the mode $\phi_{\mathbf{k}}$.

Hint: The "typical amplitude" δx of a variable x fluctuating around 0 is $\delta x = \sqrt{\langle x^2 \rangle}$.

Taking into account that for vacuum fluctuations, $\delta\phi_{\mathbf{k}}$ is given in (1.12), we find that the typical amplitude of $\delta\phi_L$ is

$$\delta\phi_L \sim \sqrt{\frac{k_L^3}{\omega_{k_L}}}, \quad k_L \equiv L^{-1}. \tag{1.13}$$

We conclude that $\delta\phi_L$ diverges as L^{-1} for small $L \ll m^{-1}$ and decays as $L^{-3/2}$ for large $L \gg m^{-1}$.

Observable effects of vacuum fluctuations Quantum vacuum fluctuations have observable consequences that cannot be explained by any other known physics. The three well-known effects are the spontaneous emission of radiation by atoms, the Lamb shift, and the Casimir effect. All of them have been measured experimentally.

The spontaneous emission of a photon by a hydrogen atom in vacuum occurs as a result of the transition between the states $2p \to 1s$. This effect can only be explained if we consider the interaction of electrons with the vacuum fluctuations of the electromagnetic field. Without these fluctuations, the hydrogen atom would have remained forever in the $2p$ state.

The Lamb shift is a small difference between the energies of the $2p$ and $2s$ states of the hydrogen atom. This shift occurs because the electron "clouds" have different geometrical shapes for the $2p$ and $2s$ states and hence interact differently with vacuum fluctuations of the electromagnetic field. The measured energy

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difference, corresponding to the frequency ≈ 1057MHz, is in a good agreement with the theoretical prediction.

The Casimir effect is manifested as an attractive force between two parallel *uncharged* conducting plates. The force decays with the distance L between the plates as $F \sim L^{-4}$. This effect is explained by considering the shift of the energy of zero-point fluctuations of the electromagnetic field due to the presence of the conductors.

1.5 Particle interpretation of quantum fields

The classical concept of particles involves point-like objects moving along specific trajectories. Experiments show that this concept does not actually apply on subatomic scales. For an adequate description of photons and electrons and other elementary particles, one needs to use a relativistic quantum field theory (QFT) in which the basic objects are not particles but quantum fields. For instance, the quantum theory of photons and electrons (quantum electrodynamics) describes the interaction of the electromagnetic field with the electron field. Quantum states of the fields are interpreted in terms of corresponding particles. Experiments are then described by computing probabilities for specific field configurations.

The energy levels of a "quantum oscillator $\phi_{\mathbf{k}}$" are $E_{n,\mathbf{k}} = \left(\frac{1}{2}+n\right)\omega_k$ where $n = 0, 1, \ldots$ At level n the energy $E_{n,\mathbf{k}}$ is greater than the ground state energy by $\Delta E = n\omega_k = n\sqrt{k^2+m^2}$, which is equal to the energy of n relativistic particles of mass m with momentum $\mathbf{k}$. Therefore the excited state with energy $E_{n,\mathbf{k}}$ is interpreted as describing n particles of momentum $\mathbf{k}$. We refer to such states as having the *occupation number* n.

A classical field corresponds to states with large occupation numbers, $n \gg 1$. In this case, quantum fluctuations can be very small compared to expectation values of the field.

A free, noninteracting field with given occupation numbers will remain in the corresponding state forever. On the other hand, in an interacting field occupation numbers can change with time. An increase in the occupation number for a mode $\mathbf{k}$ is interpreted as production of particles with momentum $\mathbf{k}$.

1.6 Quantum field theory in classical backgrounds

"Traditional" QFT deals with problems of finding cross-sections for transitions between different particle states, such as scattering of one particle on another. For instance, typical problems of quantum electrodynamics are:

(i) Given the initial state (at time $t \to -\infty$) of an electron with momentum $\mathbf{k}_1$ and a photon with momentum $\mathbf{k}_2$, find the cross-section for the scattering into the final state (at $t \to +\infty$) where the electron has momentum $\mathbf{k}_3$ and the photon has momentum $\mathbf{k}_4$.

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This problem is formulated in terms of quantum fields in the following manner. Suppose that ψ is the field representing electrons. The initial configuration is translated into a state of the mode $\psi_{\mathbf{k}_1}$ with the occupation number 1 and all other modes of the field ψ having zero occupation numbers. The initial configuration of "oscillators" of the electromagnetic field is analogous – only the mode with momentum $\mathbf{k}_2$ is occupied. The final configuration is similarly translated into the language of field modes.

(ii) Initially there is an electron and a positron with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$. Find the cross-section for their annihilation with the emission of two photons with momenta $\mathbf{k}_3$ and $\mathbf{k}_4$.

These problems are solved by applying perturbation theory to a system of infinitely many weakly interacting quantum oscillators. The required calculations are usually rather tedious because of the vacuum polarization effects which are due to the couplings of the excited "oscillators" with infinitely many "oscillators" in the ground state.

In this book we study quantum fields interacting only with a strong external field called the *background*. It is assumed that the background field is adequately described by a classical theory and does not need to be quantized. In other words, our subject is *quantum fields in classical backgrounds*. A significant simplification comes from considering quantum fields that interact *only* with classical backgrounds but not with other quantum fields. Such quantum fields are also called *free* fields, even though they are coupled to the background.

Typical problems of interest to us are:

(i) Computation of probabilities for transitions between various configurations of quantum field under the influence of a classical background field, which describe the process of particle production by the external field.

(ii) Determination of the energy level shifts for the quantum fluctuations due to the presence of the background. Since the vacuum contribution to gravity is assumed to have been subtracted already, it is likely that these energy shifts contribute to gravity.

(iii) Calculation of the backreaction of a quantum field on the classical background. For example, the external gravitational field influences the vacuum fluctuations shifting their zero-point energy levels. As a result, the vacuum fluctuations begin to contribute to a gravitational field. Their contribution can be described by an effective energy-momentum tensor, which is determined by the strength of the external gravitational field.

1.7 Examples of particle creation

A quantum oscillator in an external classical field A nonstationary gravitational background influences quantum fields in such a way that the frequencies

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ω_k become time-dependent, $\omega_k(t)$. We shall examine this situation in detail in Chapter 6. For now, we simplify our task and consider the behavior of a single harmonic oscillator with a time-dependent frequency $\omega(t)$. The energy of such an oscillator is not conserved and the oscillator exhibits transitions between different energy levels.

Let us assume that an oscillator satisfies the following equation of motion:

$$\ddot{q}(t) + \omega_0^2 q(t) = 0, \text{ for } t < 0 \text{ and } t > T; \tag{1.14}$$
$$\ddot{q}(t) - \Omega_0^2 q(t) = 0, \text{ for } 0 < t < T,$$

where ω_0 and Ω_0 are real constants.

Exercise 1.3
Given the solution of equation (1.14), $q(t) = q_1 \sin \omega_0 t$ for $t < 0$, and assuming that $\Omega_0 T \gg 1$ verify that for $t > T$

$$q(t) = q_2 \sin(\omega_0 t + \alpha),$$

where α is a constant and

$$q_2 \approx \frac{1}{2} q_1 \sqrt{1 + \frac{\omega_0^2}{\Omega_0^2}} \exp(\Omega_0 T). \tag{1.15}$$

It follows from (1.15) that the oscillator has a large amplitude, $q_2 \gg q_1$, for $t > T$. Thus the final state has much larger energy than the initial state and it can be then interpreted as a state with many particles produced within the time interval $T > t > 0$.

Exercise 1.4
Estimate the number of particles produced, assuming that the oscillator is initially in the ground state.

The Schwinger effect A static electric field can create electron–positron (e^+e^-) pairs. This effect, called the *Schwinger effect*, is currently on the verge of being experimentally verified.

To understand the Schwinger effect qualitatively, we may imagine a virtual e^+e^- pair in a constant electric field of strength E. If the particles move apart from each other a distance l, they receive energy leE from the electric field. In the case when this energy exceeds the rest mass of the two particles, $leE \geq 2m_e$, the pair becomes real and the particles continue to move apart. The typical separation of the virtual pair is of order the Compton wavelength $2\pi/m_e$. More precisely, the probability of separation by a distance l turns out to be $P \sim \exp(-\pi m_e l)$. Therefore the probability of creating an e^+e^- pair is

$$P \sim \exp\left(-\frac{m_e^2}{eE}\right). \tag{1.16}$$

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