1 The formation and analysis of optical waveguides

1.1 Introduction to optical waveguides

Optical waveguides are made from material structures that have a core region which has a higher index of refraction than the surrounding regions. Guided electromagnetic waves propagate in and around the core. The transverse dimensions of the core are comparable to or smaller than the optical wavelength. Figure 1.1(a) illustrates a typical planar waveguide. Figure 1.1(b) illustrates a typical channel waveguide. For rigorous electromagnetic analysis of such guided-wave structures, Maxwell's vector equations should be used. Many of the theoretical methods used in the analysis of optical guided waves are very similar to those used in microwave analysis. For example, modal analysis is again a powerful mathematical tool for analyzing many devices, applications and systems.

However, there are also important differences between optical and microwave waveguides. In microwaves, we usually have closed waveguides inside metallic boundaries. Metals are considered as perfect conductors at most microwave frequencies. Microwaves propagate within the metallic enclosure. Figure 1.2 illustrates a typical microwave rectangular waveguide. In these closed structures, we have only a discrete set of waveguide modes whose electric fields terminate at the metallic boundary. Microwave radiation in the waveguide may be excited either by an electric field or by a current loop. At optical wavelengths, we avoid the use of metallic boundaries because of their strong absorption of radiation. Ideal optical waveguides, such as those illustrated in Fig. 1.1(a) and (b), are considered to have dielectric boundaries extending to infinity. They are called open waveguides. Optical guided-wave modes are waves trapped in and around the core. They can be excited only by electric fields.

1.1.1 Differences between optical and microwave waveguides

Mathematically, modes represent propagating homogeneous[1] solutions of Maxwell's electromagnetic equations in waveguide structures that have constant cross-section and infinite length. Homogeneous solutions means that these are the propagating electric and magnetic fields that satisfy the differential equations and all the boundary conditions in the absence of any radiation source.[2] There are three important differences between optical and microwave waveguide modes and their utilization.

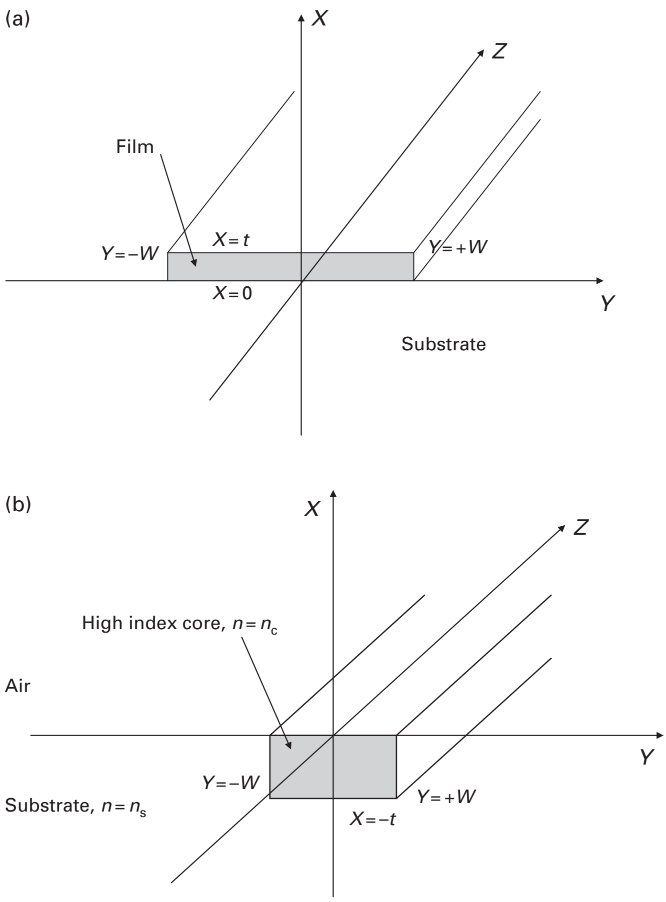

Fig. 1.1. An optical waveguide. (a) A planar waveguide. The substrate and the film are so wide in the Y direction that W can be approximated by ∞. The substrate thickness is also considered to be ∞ in the $-x$ direction. Guided-wave modes could propagate in any direction in the YZ plane. (b) A channel waveguide. The high index core ($-t \leq x \leq 0$, $-W \leq y \leq +W$) is embedded in the substrate. The core is very long in the z direction with $n_c > n_s > 1$. The guided wave propagates in the z direction.

(1) In open dielectric waveguides, the discrete optical modes have an evanescent field outside the core region (the core is often called vaguely the optical waveguide). There may be a significant amount of energy carried in the evanescent tail. The evanescent field may be used to achieve mutual interactions with the fields of other modes of such waveguides or structures. The evanescent field interaction is very important in devices such as the dielectric grating filter, the distributed feedback laser and the directional coupler.

(2) The mathematical analysis is more complex for open than for closed waveguides. In fact, there exists no analytical solution of three-dimensional open channel waveguide modes (except the modes of the round step index fiber) in the closed form. One must use either numerical analysis or approximate solutions in order to find the field distribution of optical channel waveguide modes.

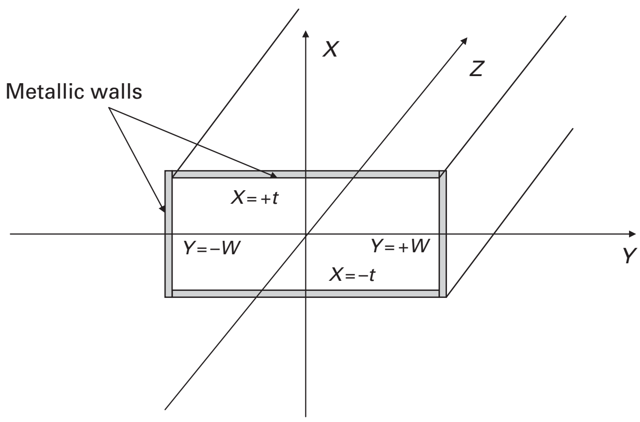

Fig. 1.2. A microwave waveguide. The rectangular waveguide has metallic walls at $y = \pm W$ and at $x = \pm t$. Guided waves propagate along the Z direction in the hollow region, $-t < x < +t$, $-W < y < +W$.

(3) In addition to the set of guided modes that have discrete eigenvalues, there is an infinite set of continuous modes in open waveguides. Only the sum of the discrete and continuous modes constitutes a complete set of orthogonal functions. It means that, rigorously, any arbitrary incident field should be expanded mathematically as a summation of this complete set of modes. At any dielectric discontinuity, the boundary conditions of the continuity of electric and magnetic fields are satisfied by the summation of both the guided-wave modes and the continuous modes on both sides of the boundary. In other words, continuous modes are excited at any discontinuity. Energy in the continuous modes is radiated away from the discontinuity. Thus, continuous modes are called radiation modes.

1.1.2 Diffraction of plane waves in waveguides

The propagation and properties of optical waves in optical waveguides can also be understood from conventional optical analysis of plane wave propagation in multilayered media. A typical optical planar waveguide is illustrated in Fig. 1.3. It has a high index film surrounded by cladding and a substrate; both have a lower index of refraction. The width of the film, the cladding and the substrate, extend to $y = \pm\infty$. The thickness of the substrate and cladding also extends to infinity in the x direction. If we analyze optical plane waves propagating in multilayered media such as that shown in Fig. 1.3, we find that there are three typical cases.

(1) In the first of these, a plane wave is incident obliquely on the film from either $x << 0$ or $x >> t$. Without any loss of generality, let us assume that the plane wave is polarized in the y direction. It propagates in the xz plane in a direction which makes an angle θ_j with respect to the x axis. The angle, θ_j, will be different in different layers, where j designates the layer with index n_j. For example, plane waves in the film with index n_1 will have a functional form, $\exp(\pm\, \mathrm{j} n_1 k \sin\theta_1 z)\exp(\pm\, \mathrm{j} n_1 k \cos\theta_1 x)\exp(\mathrm{j}\omega t)$.

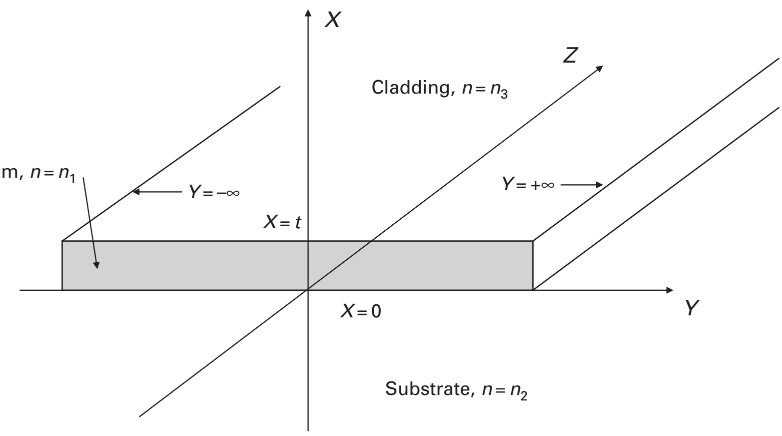

Fig. 1.3. The index profile in a planar waveguide.

There will be reflected and transmitted waves at the top and bottom boundaries of the film. The continuity of the tangential electric field demands that $n_1 k \sin\theta_1 = n_2 k \sin\theta_2 = n_3 k \sin\theta_3$ at the boundaries. There is a continuous range of real values of θ_j that will satisfy Maxwell's equations and the boundary conditions in all the layers. Plane waves with real values of θ_j represent radiation waves for $x < 0$ and for $x > t$ because they propagate in the x direction. In the language of modal analysis, the multiple reflected and refracted waves constitute the radiation modes with continuous eigenvalues β_{xj} ($\beta_{xj} = n_j k \cos\theta_j$) in the x direction, and β_{xj} is real.

(2) In the second cases, the y-polarized plane waves are trapped in the high index film by total internal reflections from the top and the bottom boundaries of the film at $x = 0$ and $x = t$. In this case the plane waves in the film will still have the functional variation of $\exp(\pm\, \mathrm{j} n_1 k \ \sin\theta_1\, z) \exp(\pm\, \mathrm{j} n_1 k \ \cos\theta_1\, x) \exp(\mathrm{j}\omega\, t)$ with real values of θ_1. When θ_1 is sufficiently large, total internal reflection occurs at the boundaries. In total internal reflection, "$n_1 k \sin\theta_1$" is larger than $n_j k$ of the surrounding media, and θ_j (for $j \neq 1$) becomes imaginary in order to satisfy the boundary conditions at all values of z. The fields in the cladding and substrate regions, $x < 0$ and $x > t$, decay exponentially away from the boundaries. When the trapped waves in the high index film are bounced back and forth between the two boundaries, they will cancel each other because of the difference in phase and yield zero total field except at specific values of θ_1 at which the round trip phase shift of the reflections is a multiple of 2π. In other words, trapped waves can only have discrete values of real propagation constant, β_{x1} ($\beta_{x1} = n_1 k \cos\theta_1$), in the film in the x direction. It means that plane waves in the substrate and cladding (or air) only have discrete imaginary θ_j values outside the film. As we shall show later, *the non-zero (i.e. the homogeneous solutions of wave equations) waves trapped in the high index film at these specific θ_1 values constitute the various orders of guided waves. Each order of guided wave propagates in the z direction with a phase velocity equal to ω/β_1* ($\beta_1 = n_1 k \sin\theta_1$).

(3) Let us assume that, in the third situation, the index of the substrate n_2 is higher than the index of the cladding n_3, and lower than the index of the film n_1. In this case, there are propagating plane waves in two regions of x: in the substrate and in the high index film region. The value of θ_1 is just large enough so that plane waves are totally internally reflected at the boundary between the film and the top cladding. Only the field in the cladding region now decays exponentially away from the film boundary.

When there are also index variations in the lateral direction (i.e. the y direction) similar observations, like those we discussed in (2), can be made for optical planar guided waves propagating in the lateral direction in the yz plane. Guided-wave modes in a channel waveguide such as the one shown in Fig. 1.1(b) can be analyzed as planar guided-wave modes totally internal reflected at the lateral boundary at $y = \pm W$, see Section 1.2.6. There will be evanescent fields in the y direction at $y > W$ and $y < -W$.

1.1.3 General characteristics of guided waves

In summary, optical waveguides always have a higher index core, surrounded by lower index regions, so that optical guided waves in the core can be considered as waves trapped in the core with evanescent field in the surrounding regions. There are also radiation waves (or cladding waves) that also propagate in the structure. The field distribution and the propagation constant of the guided waves are controlled by the transverse dimensions of the core and the refractive indices of the core and all the surrounding regions. In order to understand more clearly the properties of modes in the optical waveguide, electromagnetic analysis of modes in optical waveguides is presented in the next section.

The most important characteristics of guided-wave modes are the exponential decay of their evanescent tails, the distinct polarization associated with each mode, and the excitation of continuous modes at any defect or dielectric discontinuity that causes diffraction loss of the guided-wave mode. The evanescent tail ensures that there is only minor perturbation of the mode pattern for structure changes several decay lengths away from the surface of the high index layer.

Since propagation loss of the guided-wave modes is caused usually by scattering or absorption, the attenuation rate of the guided mode will be very low as long as there is very little absorption or scattering loss in or near the high index layer. The most common causes of absorption loss are the placement of a metallic electrode nearby, the absorption of the core material, and the use of semiconductor cladding or substrate (or core and cladding) that has absorption. In electro-absorption modulators or switches (discussed in Section 3.2) the absorption of the waveguide is controlled by an electrical signal so that the output optical power is modulated by the electrical signal. Besides absorption, the propagation losses are most commonly caused by volume scattering in the layers or by surface scattering at the dielectric interfaces. Volume scattering is introduced by defects in the material developed during growth or processing deposition. Surface scattering is created usually through roughness incurred in the fabrication processes such as etching and lift-off. Scattering converts the energy in the guided-wave mode into radiation modes.

The exponential decay rate of any guided-wave mode in the media surrounding the core is determined only by the indices of the layers (e.g. either the cladding index at $x > t$ or the substrate index at $x < 0$, in planar waveguides) and the β value of the mode. The $\beta c/\omega$ value is called the effective index, n_{eff}, of the mode. The velocity of light in free space c divided by the effective index is the phase velocity of the guided-wave mode in the z direction. For the same polarization, lower order modes will have a larger effective index (i.e. larger β) and faster exponential decay outside the core. For the same defects or interface roughness, modes that have a smaller effective index will be scattered more strongly into radiation modes. Therefore, higher order modes usually have larger attenuation. Any mode that has an effective index very close to the refractive index of the substrate or cladding will have large scattering loss. It is called a weakly propagating mode.

On the other hand, the evanescent tail also enables us to affect the propagation of the guided-wave mode by placing perturbations adjacent to the core of the high index layer. For example, in the next chapter, we will discuss the directional coupler formed by two waveguides placed adjacent to each other or by a grating filter fabricated on top of a waveguide.

1.2 Electromagnetic analysis of modes in optical waveguides

In order to understand clearly the electromagnetic properties of guided waves, modal analysis of an optical waveguide is presented in this section. The rigorous mathematical analysis of simple planar waveguides such as those shown in Fig. 1.1(a) will be presented first. In principle, modes of planar waveguides (or a summation of planar guided-wave modes) may propagate in any direction in the plane of the waveguide (i.e. the yz plane). However, for simplicity and without any loss of generality, the mathematical solution of the modes of the planar waveguide will be presented first just for modes propagating in the z direction. How these modes of planar waveguide (or combination of modes) propagate in any arbitrary direction in the yz plane will be discussed in terms of these z-propagating modes.

The geometry of channel waveguides is usually too complex for us to find mathematically the solutions of the Maxwell's equations in closed form. Numerical simulation programs such as *Rsoft BeamProp©* are used. The exception is the solution of the circular symmetric modes in step-index round fibers. The modes of optical fibers have been discussed in many books [1]. They will not be repeated here. We will discuss in Section 1.2.6 an approximate analysis, called the effective index analysis, of the modes of open rectangular channel waveguides such as those shown in Fig. 1.1(b). Results obtained from the effective index analysis are accurate only for well-guided modes, i.e. modes with a short evanescent tail. Nevertheless, the effective index analysis enables us to understand the basic properties of all channel guided-wave modes.

It will be clear later from the discussions of planar and channel waveguide modes that the fields of most guided-wave modes can be approximated just by the dominant component of the mode perpendicular to the direction of propagation. In other words, instead of solving Maxwell's vector equations, modes of arbitrary cross-section of the core may be calculated approximately by a scalar equation in terms of just the dominant field. Such a quasi-scalar approximation of the Maxwell's equations will be presented after the discussion of planar and channel waveguide modes.

1.2.1 The asymmetric planar waveguide

A typical uniform dielectric thin film planar waveguide has been shown in Fig. 1.3, where the film, the cladding and the substrate are all uniform and infinitely wide in the y and the z directions. The film typically has a thickness of the order of a wavelength or less, supported by a substrate and a cladding many wavelengths (or infinitely) thick. The refractive index of the film (i.e. the waveguide core), n_1, is higher than the indices of the surrounding layers.

Since the structure is identical in any direction in the yz plane, we will temporarily choose the $+z$ axis as the direction of propagation in our mathematical analysis. For planar modes, we further assume $\partial/\partial y \equiv 0$. This assumption is similar to the assumption made for plane waves in a homogeneous medium in many textbooks. This assumption on the y variation applies in Sections 1.2.2, 1.2.3 and 1.2.4.

1.2.2 TE and TM modes in planar waveguides

The variation of the refractive index in the transverse direction is independent of z in Fig. 1.3. From discussions of electromagnetic theory in classical electrical engineering textbooks, we know that modes for structures that have constant transverse cross-section in the direction of propagation can be divided into TE (transverse electric) and TM (transverse magnetic) types. Note that TE means that there is no electric field component in the direction of propagation, TM means that there is no magnetic field component in the direction of propagation.

For planar waveguides, if we substitute $\partial/\partial y = 0$ into $\nabla\times\underline{E}$ and $\nabla\times\underline{H}$ in Maxwell's equations, we obtain two separate groups of equations:

$$\frac{\partial E_y}{\partial z} = \mu\,\partial H_x/\partial t,\ \frac{\partial E_y}{\partial x} = -\mu\partial H_z/\partial t,\ \frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z} = -\varepsilon\partial E_y/\partial t,$$

and

$$\frac{\partial H_y}{\partial z} = -\varepsilon\partial E_x/\partial t,\ \frac{\partial H_y}{\partial x} = \varepsilon\partial E_z/\partial t,\ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = \mu\partial H_y/\partial t. \tag{1.1}$$

Clearly, E_y, H_x, and H_z are related only to each other, and H_y, E_x, and E_z are related only to each other. Since the direction of propagation is z, the solutions of the first group of equations are the TE modes. The solutions of the second group of equations are the TM modes. In other words, all planar waveguide modes can be divided into TE and TM types.

Since ε is only a function of x, the z variation of the fields must be the same in all layers. This is a consequence of the requirement for continuity of E_y or H_y for all z. Let us also assume that the time variation of the field is $e^{j\omega t}$. Then, for propagating waves in the +*z* direction, we will have an exp(–jβz) variation, while the waves in the –*z* direction will have an exp(jβz) variation. The TE wave equations for planar E_y in Eq. (1.1) can now be written as a product of a function in *y* and a function in *z*, i.e. $E_y(x,z) = E_y(x)E_y(z)$

$$\left[\frac{\partial^2}{\partial x^2} + \left(\omega^2\mu\varepsilon(x) - \beta^2\right)\right] E_y(x)E_y(z) = 0, \tag{1.2a}$$

$$\left[\frac{\partial^2}{\partial z^2} + \beta^2\right] E_y(z) = 0, \tag{1.2b}$$

or

$$\left[\frac{\partial^2}{\partial x^2} + \left(\omega^2\mu\varepsilon(x) - \beta^2\right)\right] E_y(x) = 0. \tag{1.2c}$$

Similar equations exist for TM modes.

1.2.3 TE modes of planar waveguides

The planar TE modes (i.e. modes with $\partial/\partial y = 0$) in the planar waveguides are eigen solutions of the equation,

$$\begin{aligned}
\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2\mu\varepsilon(x)\right] E_y(x,z) &= 0 \\
\varepsilon(x) &= n_3^2\varepsilon_o \quad x \geq t \\
&= n_1^2\varepsilon_o \quad t > x > 0 \\
&= n_2^2\varepsilon_o \quad 0 \geq x \\
H_x = -\frac{j}{\omega\mu}\frac{\partial E_y}{\partial z}, \quad H_z &= \frac{j}{\omega\mu}\frac{\partial E_y}{\partial x}.
\end{aligned} \tag{1.3}$$

Here, ε_o is the free space electric permittivity. All layers have the same magnetic permeability μ, and the time variation is exp(jωt). Note that when E_y is known, H_x and H_y can be calculated directly from E_y. The boundary conditions are the continuity of the tangential electric and magnetic fields at $x = 0$ and at $x = t$. As we shall see in the following subsections, the TE modes can be further classified into three sub-groups. One group, the guided waves, is characterized as plane waves trapped inside the film, and the other two groups are two different kinds of combination of radiating plane waves known as substrate modes and air modes. Mathematically, all the TE modes form a complete set of eigenfunctions, meaning that any arbitrary electric field polarized in the *y* direction with $\partial/\partial y = 0$ can be expanded as a summation of TE modes.

1.2.3.1 TE planar guided-wave modes

Mathematically, Eq. (1.2) and (1.3) suggest that the solution of $E_y(x)$ is either a sinusoidal or an exponential function, and the solution of $E_y(z)$ is $e^{\pm j\beta z}$. Guided by the discussion in

Section 1.1, we look for solutions of $E_y(x)$ with sinusoidal variations for $t > x > 0$ and with decaying exponential variations for $x > t$ and $x < 0$. Since we have chosen the time variation as $e^{+j\omega t}$, the $\exp(-j\beta z)$ variation of $E_y(z)$ represents a forward propagating wave in the $+z$ direction. In short, we will assume the following functional form for $E_y(x,z)$:

$$\begin{aligned} E_m(x,z) &= A_m \sin(h_m t + \phi_m)\exp[-p_m(x-t)]\exp(-j\beta_m z) && x \geq t \\ E_m(x,z) &= A_m \sin(h_m x + \phi_m)\exp(-j\beta_m z) && t > x > 0 \\ E_m(x,z) &= A_m \sin\phi_m \exp[q_m x]\exp(-j\beta_m z), && 0 \geq x \end{aligned}$$

where in order to satisfy Eq. (1.2a, b and c)

$$\begin{aligned} (\beta_m/k)^2 - (p_m/k)^2 &= n_3^2 \\ (\beta_m/k)^2 + (h_m/k)^2 &= n_1^2 \\ (\beta_m/k)^2 - (q_m/k)^2 &= n_2^2. \end{aligned} \tag{1.4}$$

The subscript m stands for the mth order solution of Eq. (1.3). Equation (1.3) is clearly satisfied by E_m in all the individual regions. We have also chosen this functional form so that the continuity of E_y is automatically satisfied at $x = 0$ and $x = t$. In order to satisfy the magnetic boundary conditions[3] at $x = 0$ and $x = t$, h_m, q_m, and p_m must be the mth set of the roots of the transcendental equations which are also called the characteristic equations,

$$\tan[(h_m/k)kt + \phi_m] = -h_m/p_m \quad \text{and} \quad \tan\phi_m = h_m/q_m. \tag{1.5}$$

For a given normalized thickness kt, there are only a finite number of roots of the characteristic equations yielding a discrete set of real values for h, p, and q. For this reason, the guided-wave modes are also called the discrete modes. They are labeled by the integer subscript m ($m = 0, 1, 2,\ldots$). The lowest order mode with $m = 0$ has the largest β value, $\beta_0 > \beta_1 > \beta_2 > \beta_3 \ldots$ and $h_0 < h_1 < h_2 \ldots$. Moreover, one can show that the number of times in which $\sin(h_m x + \phi_m)$ is zero is m. The H_x and H_z fields can be calculated from E_y according to Eq. (1.1). Since $\beta_m >> h_m$, H_x is the dominant magnetic field for TE modes. The mth TE mode propagating in the $-z$ direction will have $e^{j\beta z}$ variation for $E_y(z)$, with the same xy field variation given in Eq. (1.4).

The exponential decay rate of any guided-wave mode is determined only by the index of the surrounding layer (either at $x > t$ or at $x < 0$) and the β/k value of the mode. The β/k value is called the effective index, n_{eff}, of the mode. The velocity of light in free space divided by effective index n_{eff} is the phase velocity of the guided-wave mode. For the same polarization, lower order modes will have larger effective index and faster exponential decay. For the same $\Delta\varepsilon$ of defects or interface roughness, modes that have a smaller effective index will be scattered more strongly into radiation modes, i.e. substrate and air modes. Therefore, higher order modes usually have larger attenuation.

1.2.3.2 TE planar guided-wave mode in a symmetrical waveguide

In order to visualize why there should be only a finite number of modes, let us consider the example of a symmetrical waveguide. In that case, $n_2 = n_3 = n$ and $p_m = q_m$. The quadratic equations for h_m and β_m and the transcendental equation now become

$$\left(\frac{h_m}{k}\right)^2 + \left(\frac{p_m}{k}\right)^2 = n_1^2 - n^2, \tag{1.6}$$

and

$$\tan\left[\left(\frac{h_m}{k}\right)kt\right] = \frac{-2\frac{h_m}{p_m}}{1 - \frac{h_m{}^2}{p_m{}^2}}. \tag{1.7}$$

Since,

$$\tan\left[2\left(\frac{h_m}{k}\right)\frac{kt}{2}\right] = \frac{2\,\tan\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right]}{1 - \tan^2\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right]},$$

Eq. (1.7) can be reduced to two equations,

$$\tan\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right] = \frac{p_m/k}{h_m/k}, \quad \text{hence} \quad \frac{h_m}{k}\tan\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right] = \frac{p_m}{k}, \tag{1.8a}$$

or

$$\tan\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right] = -\frac{h_m/k}{p_m/k}, \quad \text{hence} \quad -\frac{h_m}{k}\cot\left[\left(\frac{h_m}{k}\right)\frac{kt}{2}\right] = \frac{p_m}{k}. \tag{1.8b}$$

In the coordinate system of p_m/k and h_m/k, the solutions of Eq. (1.6) and (1.7) are given by the intersections of the two curves representing the quadratic equation, $(h_m/k)^2 + (p_m/k)^2 = n_1^2 - n^2$, and one of the two equivalent tangent equations, (1.8a) or (1.8b). To summarize, there are two sets of equations. The solutions for the first tangent equation (1.8a) and the quadratic equation (1.6) are known as the even modes because they lead to field distributions close to a cosine variation in the film. They are symmetric with respect to $x = t/2$. The solutions from the second tangent equation (1.8b) and the quadratic equation (1.6) are called the odd modes because the fields in the film have distributions close to sine variations. They are anti-symmetric with respect to $x = t/2$.

Let us examine the even modes in detail. If we plot the quadratic equation of h_m/k and p_m/k, it is a circle with a radius $(n_1^2 - n^2)^{1/2}$. The curve describing the first tangent equation will be obtained from those values of h_m/k and p_m/k whenever the left hand side (LHS) is equal to the right hand side (RHS) of the tangent equation. The RHS is just p_m/k. The LHS has a tangent which is a multi-valued function. It starts from 0 whenever $(h_m/k)kt/2$ is 0, π, or $m\pi$. It approaches + or − infinity when $(h_m/k)kt/2$ approaches $+\pi/2$ or $-\pi/2$, or $(m+\pi/2)$ or $(m-\pi/2)$ where m is an integer. The curves representing these two equations are illustrated in Fig. 1.4. Clearly there is always a solution as long as $n_1 > n$, i.e. there is an intersection of the two curves, no matter how large (or how small) is the circle (i.e. the n_1 value). This is the fundamental mode, labeled by $m = 0$. However, whether there will be an $m \geq 1$ solution depends on whether the radius is larger than $2\pi/kt$. There will be $m = j$ solutions when the radius is larger than $2j\pi/kt$. Notice that $h_0 < h_1 < h_2 \ldots$ and $\beta_0 > \beta_1 > \beta_2 > \ldots$. When the radius of the circle is just equal to $2j\pi/kt$, the value for p/k is 0. This is the cut-off point for the jth ($j > 1$) mode.