Elementary Probability for Applications

This book is an introduction to probability theory concentrating on the results that are the most useful for applications. Its philosophy is that the best way to learn probability is to see it in action and so there are more than 350 problems and 200 examples. The examples contain several classics such as the birthday problem and Monty Hall, but they also include a number of applications not found in other books.

Rick Durrett received his Ph.D. in operations research from Stanford University in 1976. After 9 years at UCLA and 25 at Cornell University, he came to Duke University in 2010, where he is a James B. Duke Professor in the math department. He is the author of 8 books and more than 170 journal articles on a wide variety of topics, and he has supervised more than 40 Ph.D. students. He is a member of the National Academy of Science and the American Academy of Arts and Sciences and a Fellow of the Institute of Mathematical Statistics.

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CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521867566

First published 2009
Reprinted 2010 (twice), 2012
4th printing 2022

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data
Durrett, Richard, 1951–
Elementary probability for applications / Rick Durrett.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-521-86756-6 (hardback)
1. Probabilities. I. Title.
QA273.D8638 2009
519.2–dc22 2009020688

ISBN 978-0-521-86756-6 Hardback

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Preface

> Probability is the most important concept in modern science especially as nobody has the slightest notion what it means.
>
> Bertrand Russell

About 15 years ago, I wrote the book *Essentials of Probability*, which was designed for a one-semester course in probability that was taken by math majors and students from other departments. This book is, in some sense, the second edition of that book, but there are several important changes:

- Chapter 1 quickly introduces the notions of independence, distribution, and expected value, which previously made their entrance in Chapters 2, 3, and 4. This makes it easier to discuss examples; for example, we can now talk about the expected value of bets.
- For 5 years these notes were used in a course for students who knew only a little calculus and were looking to satisfy their distribution requirement in mathematics, so it is aimed at a wider audience.
- Markov chains are covered, and thanks to a suggestion of Lea Popovic, this topic appears right after the notion of conditional probability is discussed. This material is usually covered in an undergraduate stochastic processes course, if you are fortunate enough to offer one in your department, but in our experience this material is popular with students.
- Continuous distributions are presented as an optional topic. This decision originated to minimize the reliance on calculus, but in time I have grown to enjoy abandoning the boring mechanics of marginal and conditional distributions to spend more time talking about probability.

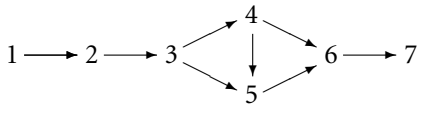
This book, like its predecessor, takes the philosophy that the best way to learn probability is to see it in action. There are more than 350 problems and 200 examples. These contain all the old standards: the World Series, dice and card games, birthday problem, Monty Hall, medical testing posterior probabilities, and various applications of the central limit theorem. However, it also contains a number of topics that are rarely covered: Benford's law, the TV show

Deal or No Deal, alliteration in Shakespeare, Wayne Gretzky's scoring record, lottery double winners, the trials of O. J. Simpson and Sally Clark, how to play blackjack, cognitive dissonance in monkeys, the hot hand in basketball, and option pricing.

The discussion in the text follows the philosophy of *Dragnet's* Joe Friday: "Just the facts, ma'am." However, there are several dozen figures that show the shapes of distributions and help to explain the arguments. Copyright laws prevent me from calling this book "Probability for Dummies," but that would be a misnomer. The first two words of the title are "Elementary Probability," which means that there are few formal prerequisites. I have done my best to explain things clearly, but the reader (and the instructor) should be warned that thinking is required.

How should you teach from this book?

The easiest way to explain this is with a flowchart of the chapters:



Chapter 1 introduces the language we need to talk about examples. Chapter 2 covers combinatorial probability and is followed by Chapter 3's treatment of conditional probability. At this point, one can go on to Markov chains in Chapter 4 (which I prefer) or to continuous distributions in Chapter 5. From Chapter 4 you can go to Chapter 5 or leave this boring topic to the instructor of the statistics course that follows yours and go on to the law of large numbers and central limit theorem in Chapter 6. The all-important normal is a continuous distribution, of course, but all computations for it are done with tables, so the only concept one needs is the distribution function. Finally, Chapter 7 is a brief introduction to option pricing. I find this makes a nice final lecture before one turns to the business of reviewing material in preparation for the final exam.

Supporting cast

The writing of this book benefited from the comments of several people paid by Cambridge University Press to read various chapters, and in particular by the efforts of one reader who made hundreds of comments on the writing style. In the spring quarter of 2008, Ed Waymire used the book at Oregon State, and in the fall quarter of 2008, Michael Phelan used the book at U. C. Irvine. I am grateful to Michael for his many comments and his enthusiasm for the book.

When *Essentials of Probability* was written, my sons David and Greg were 6 and 4, and Tipper Gore was complaining about the lack of values in the music I listened to. Now David is a senior at Ithaca College, one semester away from graduating with a major in journalism, and wondering if the economic collapse brought on by 8 years of the Bush administration will keep him from getting a job. Greg, who is a junior at MIT double majoring in computer science and math, has better long-term job prospects, but he will probably go to graduate school before deciding how close he wants to be to the real world.

This brings me to the two women who are the most important for this book. The first is my wife, Susan. After 28 years of marriage and almost a dozen prefaces, I have run out of clever things to say. When the kids are home from college, as they are now during winter break, she is a flurry of activity. In between, she fills her empty nest with the *New York Times*, its crossword puzzles (an addiction I share), and tending to her parents who moved to Ithaca from the Sacramento area about 5 years ago. In December, we had our first vacation away together since David was born in 1988. Before you say "how romantic," I should admit that the trip consisted of 2 days on the beach at Half Moon Bay and 3 days of work in Berkeley.

The other important woman is my editor Lauren Cowles. After seeing *Essentials of Probability* moved around from Wadsworth to Duxbury Press and on to International Thompson Publishing and then go out of print without my being told, it is nice to be in the hands of someone who cares about my books. Even though I (and others) have spent a lot of effort debugging the book, it is inevitable that there will be typos.

In this fourth printing of the first edition, a few dozen errors have been corrected. Email lingering problems to rtd@math.duke.edu and look for lists at www.math.duke.edu/∼rtd, where you can find information about my research and other books.

Rick Durrett