Part I

Discourse on Thinking

1 Puzzling about (Mathematical) Thinking

> One . . . fact must astonish us, or rather would astonish us if we were not too much accustomed to it: How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds, how does it happen that there are so many people who are entirely impervious to it?
>
> Henri Poincaré[1]

Full of puzzles, mathematics is a puzzle in itself. Anybody who knows anything about it is likely to have questions to ask. Most of us marvel about how abstract mathematics is and wonder how one can come to grips with anything as complex and as detached from anything tangible as this. The concern of those who do manage the complexity, as did the French mathematician and philosopher of science Henri Poincaré, is just the opposite: The fortunate few who "speak mathematics" as effortlessly as they converse in their mother tongue have a hard time understanding other people's difficulty. From a certain point in our lives, it seems, mathematical understanding becomes an "all or nothing" phenomenon – either you have it, or you don't – and being in any of these two camps appears so natural that you are unable to imagine what it means to be in the other.

But the bafflement with regard to mathematics goes further than that. Literature about human thinking is teeming with resilient mathematics-related puzzles. Some of these puzzles are well known and have been fueling vocal debates for a long time now; some others are still waiting for broader attention. Let me instantiate both types of quandaries with a number of examples. Each of the five stories that follow begins with a brief description of a well-documented controversy and continues with additional teasing

[1] Poincaré (1952, p. 47).

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Anna Sfard
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questions that must occur to us the moment we manage to see a familiar situation in unfamiliar light. No solutions will be proposed at this time, and when the chapter ends, some readers may feel left midair, and rather annoyingly so. May I thus ask for your patience: Grappling with the conundrums that follow is going to take this whole book. In this chapter, my aims are to present the maladies of the present research on thinking and prepare the ground for diagnosing their sources. The attempt to follow with a cure will be made in the remaining chapters. I do hope that the long journey toward a better understanding of thinking will be not any less rewarding than the prizes that wait at its end.

1. The Quandary of *Number*

Puzzling phenomena related to mathematics can be observed already in the earliest stages in a child's development. Some of the best known and most discussed of such phenomena were first noticed and documented by the Swiss psychologist Jean Piaget.[2] To put it in Piaget's own language, young children do not *conserve number*; that is, they are not aware of the fact that mere spatial rearrangements do not change cardinality of sets of objects (or, to put it more simply, as long as nothing was added or taken away, the counting process, if repeated, always ends with the same number-word).

A child's awareness of the conservation of number is tested with the help of specially designed tasks. In one of such tasks the child is shown two numerically equivalent sets of counters arranged in parallel rows of equal length and density. The one-to-one correspondence of the counters is thus readily visible when the child is asked, "Which of the rows has more marbles?" In this situation, even young interviewees are reported to give the expected answer. One of the rows is then stretched so as to become longer without becoming more numerous and the child is asked the comparison question again. On the basis of their performance, most 4- and 5-year-olds are believed to be at the "preconservation" stage: When requested to compare the rows of the unequal length, even those of them who previously answered that "no row has more" now point to the one that has been stretched. This phenomenon appears particularly surprising in the view of the fact that by the age of 4 the majority of children have already mastered the art of counting up to 10 or 20.[3] Why is it that children who can count

[2] Piaget (1952).
[3] This mastery has been described by Rochelle Gelman and her colleagues (e.g., Gelman & Gallistel, 1978) as the ability to observe three principles of counting: the principle of *one-to-one correspondence*, that is, of assigning exactly one number-word to each element

properly do not turn to counting when presented with the question "Which of the two rows has more marbles?" "They do not yet conserve number" is a traditional Piagetian answer. Piaget's perplexing finding, as well as his diagnoses, led to a long series of additional studies in which 4- and 5-year-old children were presented with tasks best solved with the help of counting, such as set comparison or construction of numerically equivalent sets. All these studies confirmed at least one of Piaget's observations: Although skillful in counting, children tend to perform certain tasks with nonnumerical methods, which more often than not lead them to "nonstandard" results.

Over the last several decades these phenomena and their Piagetian interpretation generated much discussion.[4] For example, Margret Donaldson and James McGarrigle[5] speculated that children may have at least two good reasons to modify their answers after the change in the arrangement of sets, with none of these reasons translating into the young learners' "inability to conserve number." First, it seemed plausible that rather than relating the words *has more* to the cardinality of sets, the children attend to the immediately visible properties of the rows, such as length. Second, according to the rules of the learning–teaching game widely practiced both in schools and in children's homes, the very reiteration of the question may be interpreted by the young interviewees as a prompt for a change in the answer.[6]

In the attempt to have a closer look at this phenomenon, my colleague Irit Lavie and I have launched an *Incipient Numerical Thinking Study*.[7] Our "subjects" were Irit's 4-year-old daughter, Roni, and Roni's 7-months-older friend Eynat (see Figure 1.1), and our intention was to conduct an experiment similar to those described earlier: We would ask the girls to compare sets of counters. Although in the end our study led to findings not unlike those obtained by Piaget and his followers, it also became a source of new, previously unreported quandaries. One vignette from this study suffices to exemplify certain striking, previously unreported aspects of the children's performances. Episode 1.1, presented in the following, is the beginning of the first 20-minute-long conversation between the two

of the set that is being counted; the principle of *constant order*, that is, of always saying the number-words in the same linear arrangement; and the principle of *cardinality*, that is, the awareness of the fact that correct counting of the given set, if repeated, must end with the same number-word.

4 See, e.g., Mehler and Bever (1967) and McGarrigle and Donaldson (1974).

5 McGarrigle and Donaldson (1974).

6 Mehan (1979).

7 This is a longitudinal study ongoing since 2002. Eynat, whom Roni has known since birth, is a daughter of Roni's parents' friends. Both couples are well-educated professionals. The event took place in Roni's house. For a detailed report on the first part of this study see Sfard and Lavie (2005).

Figure 1.1. Roni and Eynat.

girls and Roni's mother. The event took place in Roni's house. Two sets of marbles were presented to the girls in identical closed boxes, with the marbles themselves invisible through the opaque walls.[8]

Episode 1.1. Comparing sets of marbles

Speaker	What is said	What is done
1. Mother	I brought you two boxes. Do you know what is there in the boxes?	Puts two identical closed opaque boxes, A and B, on the carpet, next to the girls.
2. Roni	Yes, marbles.	
3a. Mother	Right, there are marbles in the boxes.	
3b. Mother	I want you to tell me in which box there are more marbles.	While saying this, points to box A close to Eynat, then to box B.
3c. Eynat		Points to box A, which is closer to her.
3d. Roni		Points to box A.
4. Mother	In this one? How do you know?	Points to box A.

[8] The conversation was held in Hebrew. While translating to English, I made an effort to preserve the idiosyncrasies of the children's word use (thus expressions such as "this is the biggest than this one" and "it is more huge than that.")

5. Roni	Because this is the biggest than this one. It is the most.	While saying "than this one" points to box B, which is closer to her.
6. Mother	Eynat, how do you know?	
7. Eynat	Because . . . cause it is more huge than that.	Repeats Roni's pointing movement to box B when saying "than that."
8. Mother	Yes? This is more huge than that? Roni, what do you say?	Repeats Roni's pointing movement to box B when saying "than that."
9. Roni	That this is also more huge than this.	Repeats Roni's pointing movement to box B when saying "than that."
. . . . . .	. . . . . .	. . . . . .
10a. Mother	Do you want to open and discover? Let's open and see what there is inside. Take a look now.	
10b. Roni		Abruptly grabs box A, which is closer to Eynat and which was previously chosen as the one with more marbles.
11. Roni	1 . . . 1 . . . 1 . . . 2, 3, 4, 5, 6, 7, 8.	Opens box A and counts correctly.
12. Eynat	1, 2, 3, 4, 5, 6.	Opens box B and counts correctly.
13. Mother	So, what do you say?	
14. Roni	6.	
15. Mother	Six what? You say 6 what? What does it mean "six"? Explain.	
16. Roni	That this is too many.	
17. Mother	That this is too much? Eynat, what do you say?	
18. Eynat	That this too is a little.	
19. Mother	That it seems to you a little? Where do you think there are more marbles?	

Episode 1.1 ***(continued)***

20.	Roni	I think here.	Points to box A, which is now close to her (and in which she found eight marbles).
21.	Mother	You think here? And what do you think, Eynat?	
22.	Eynat	Also here.	

As predicted by the mother, the girls have shown full mastery of counting. In spite of this, they did not bother to count the marbles or even to open the boxes when asked to compare the invisible contents. Their immediate response was the choice of one of the closed boxes ([3c], [3d]). Not only did they make this instant move and agree in their decision, but they were also perfectly able to "justify" their action in a way that could have appeared adequate if not for the fact that the girls had no grounds for the comparative claims, such as "this is the biggest than this one" ([5]), "It is the most" ([7]), and "it is more huge than that" ([9]). If the startled mother had hoped that her interrogation about the reasons for the choices ([4], [6], [8]) would stimulate opening the boxes and counting the marbles, she was quickly disillusioned: Nothing less than the explicit request to open the boxes ([10a]) seemed to help.

By now, we are so familiar with the fact that "children who know how to count may not use counting to compare sets with respect to number"[9] that the episode may fail to surprise us, at least at the first reading. And yet, knowing what children usually *do not* do is not enough to account for what they *actually do*. Our young interviewees' insistence on deciding which box "has more marbles" without performing any explorations is a puzzle, one that has not been noted or accounted for in the previous studies. Unlike in conservation tasks, Roni and Eynat made their claims about the inequality without actually seeing the sets, so we cannot ascribe their choices to any visible differences between the objects of comparison. Neither can the children's surprising decision be seen as motivated by the rule "Repeated question means 'Change your answer!'": The girls chose one of the indistinguishable boxes already the first time round, before the parents had a chance to reiterate their request. Well, they were playing a guessing game, somebody may say. This would mean that the children knew they would have to verify their guess by counting the contents of the two boxes. However,

[9] Nunes and Bryant (1996, p. 35).

neither of them seemed inclined actually to perform such a verifying procedure, and when they eventually did, there was no sign they were concerned with the question whether the present answer matches the former direct choice. Moreover, the hypothesis of a guessing game, even if confirmed, still leaves many questions unanswered: Why were the girls in such perfect accord about their choices even though these choices seemed arbitrary? What was it that evidently made the chosen box so highly desirable? (Note that each of the girls wanted this box for herself; see for example, [10b].) Why after making the seemingly inexplicable decisions were the children able to answer the request for justification? On what grounds did they claim that what they chose is "the biggest" or "more huge"? Many different conjectures may be formulated in an attempt to respond to all these queries, but it seems that a real breakthrough in our understanding of children's number-related actions is unlikely to occur unless there is some fundamental change in our thinking about numerical thinking.

It seems that in order to come to grips with these and similar phenomena, one needs to go beyond the Piagetian frame of mind. Indeed, if there is little in the past research to help us account for what we saw in this study, it is probably because theory-guided researchers attend to nothing except for those actions of their interviewees that they have classified in advance as relevant to their study, and for the Piagetian investigator, the conversation that preceded opening of the boxes would be dismissable as mere "noise." The analysis of the remaining half of the event might even lead her to the claim that Roni and Eynat had a satisfactory command of numerical comparisons, although this is not the vision that emerges when the second part of the episode is analyzed in the context of the first.

2. The Quandary of *Abstraction* (and *Transfer*)

The most common explanation of the widespread failure in more advanced school-type mathematics is its highly *abstract* character. Abstracting, the specialty of scientists at large and of mathematicians in particular, has always been a highly valued activity, appreciated for its power to produce useful generalizations. It has been believed that if people engage in abstract thinking in spite of its difficulty, they do so because of the natural tendency of the human mind for organizing one's experience with the help of unifying patterns and structures. It may thus be surprising that the notion of abstraction has been getting bad press lately. True, the troubles did not really start today. The idea of abstraction boggled the minds of philosophers and of psychologists from the birth of their disciplines, and critical voices, pointing to abstraction-engendered conceptual dilemmas, could be heard for centuries. And yet, never before was it suggested, as it is now, that the term *abstraction* be simply removed from the discourse on learning.[10]

To get a flavor of the phenomena that shook researchers' confidence in the human propensity for abstracting, let us look at the brief episode that originates in the study of Brazilian street vendors conducted by Teresinha Nunes, Annalucia Schliemann, and David Carraher.[11] The 12-year-old child, M, selling coconuts at the price of 35 cruzeiros per unit, is approached by a customer.

> *Customer:* I'm going to take four coconuts. How much is that?
>
> *M, the child:* There will be one hundred five, plus thirty, that's one thirty-five . . . one coconut is thirty-five . . . that is . . . one forty!

Some time later, the child is asked to perform the numerical calculation $4 \cdot 35$ without any direct reference to coconuts or money.

> *Child:* Four times five is twenty, carry the two; two plus three is five, times four is twenty. [Answer written: 200]

The new result, so dramatically different from the former, may seem puzzling to anybody who knows a thing or two about mathematics. To put it in the researchers' own words, "How is it possible that children capable of solving a computational problem in the natural situation will fail to solve the same problem when it is taken out of its context?"[12] Solving "the same problem" in different situations means being able to view the two situations as, in a sense, the same, or at least as sufficiently similar to allow for application of the same algorithm. Being able to notice the sameness (or just the similarity) is the gist of abstracting, and the capacity for abstracting is said to be part and parcel of the human ability to "transfer knowledge" – to recycle old problem-solving procedures in new situations. What puzzled the implementers of the Brazilian study was the fact that this latter ability seemed to be absent in M, as well as in practically all the other young street vendors whom they interviewed.

One may try to account for these findings simply by saying that the main reason for the disparity between the Brazilian childrens' performances in the street and in school-like situations was their insufficient schooling. M's inability to cope with the abstract task is understandable in the view of

10 Lave and Wenger (1991).
11 Nunes, Schliemann, and Carraher (1993, p. 24).
12 Ibid., p. 23.

his almost complete lack of school learning. And yet, the question remains why it did not occur to the child to use in the school-like situation the very same algorithm that made him so successful in the street. This query becomes even more nagging in the view of the results of other cross-cultural and cross-situational studies, most of which indicated that people who are extremely skillful in solving everyday mathematical problems may have considerable difficulty with learning abstract equivalents of the real-life procedures. Consider, for example, the findings of the study conducted by Michael Cole and his colleagues in the 1960s in Liberia. Although the Kpelle people, whom the researchers observed, have shown great agility in operations involving quantities of rice and in money transactions, they seemed almost impervious to school mathematics. "Teachers complained that when they presented a problem like $2 + 6 = ?$ as an example in the classroom and then asked $3 + 5 = ?$ on a test, students were likely to protest that the test was unfair because it contained material not covered in the lesson."[13] Even in retrospect, Cole cannot overcome his bafflement:

> The question aroused by these observations remains with me to this day. Judged by the way they do puzzles or study for mathematics in school, the Kpelle appeared dumb; judged by their behavior in markets, taxis, and many other settings, they appeared smart (at least, smarter than one American visitor). How could people be so dumb and so smart at the same time?[14]

These findings are not unlike the results of many other cross-cultural and cross-situational studies, notably those on dairy warehouse workers,[15] on American shoppers and weight-watchers,[16] and on Nepalese shopkeepers.[17] In our own study, we have seen that a child may have difficulty putting together everyday and abstract mathematical procedures even if she has a reasonable knowledge of school mathematics. Consider, for example, two excerpts from an interview with a 12-year-old seventh grader,[18] whom I shall call Ron. In the first part of the conversation, the child was playing the role of a shop attendant and the interviewer presented herself as a client. The products were represented by cards featuring their names along with their authentic prices. The "vendor" and the "buyer" had a certain amount

[13] Cole (1996, p. 73). Compare Cole et al. (1971); Hoyles et al. (2001); Lave (1988); Scribner (1997); Scribner and Cole (1981).
[14] Cole (1996, p. 74).
[15] Scribner (1997).
[16] Lave (1988).
[17] Beach (1995).
[18] The interview was conducted in Hebrew by Liron Dekel (2003) as a part of her master's thesis.