Functional integration successfully entered physics as path integrals in the 1942 Ph.D. dissertation of Richard P. Feynman, but it made no sense at all as a mathematical definition. Cartier and DeWitt-Morette have created, in this book, a new approach to functional integration. The close collaboration between a mathematician and a physicist brings a unique perspective to this topic. The book is self-contained: mathematical ideas are introduced, developed, generalized, and applied. In the authors’ hands, functional integration is shown to be a robust, user-friendly, and multi-purpose tool that can be applied to a great variety of situations, for example systems of indistinguishable particles, caustics-analysis, superanalysis, and non-gaussian integrals. Problems in quantum field theory are also considered. In the final part the authors outline topics that can profitably be pursued using material already presented.

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Functional Integration:
Action and Symmetries

P. CARTIER AND C. DEWITT-MORETTE
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¹ An expression of L. Rosenfeld.
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My career began on October 1, 1944. My gratitude encompasses many teachers and colleagues. The list would be an exercise in name-dropping. For this book I wish to bring forth the names of those who have been my graduate students. Working with graduate students has been the most rewarding experience of my professional life. In a few years the relationship evolves from guiding a student to being guided by a promising young colleague.

Dissertations often begin with a challenging statement. When completed, a good dissertation is a wonderful document, understandable, carefully crafted, well referenced, presenting new results in a broad context.

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List of symbols, conventions, and formulary

Symbols

\[ A := B \]
\[ A \int B \]
\[ \theta \]
\[ B \Rightarrow A \]
\[ d^x l = dl/l \]
\[ \partial^x / \partial l = l \partial / \partial l \]
\[ \mathbb{R}^D, \mathbb{R}_D \]
\[ \mathbb{R}^{D \times D} \]
\[ X, X' \]
\[ \langle x', x \rangle \]
\[ (x, y) \]
\[ (\mathbb{M}^D, g) \]
\[ T \mathbb{M} \]
\[ T^* \mathbb{M} \]
\[ \mathcal{L}_X \]
\[ U^{2D}(S), U^{2D} \]
\[ \mathcal{P}_{\mu, \nu}(\mathbb{M}^D) \]
\[ U_{\mu, \nu} := U^{2D}(S) \cap \mathcal{P}_{\mu, \nu}(\mathbb{M}^D) \]
List of symbols, conventions and formulary

\[ h = h/(2\pi) \]
\[ [h] = ML^2T^{-1} \]
\[ \omega = 2\pi\nu \]
\[ t_B = -i\hbar\beta = -i\hbar/(k_BT) \]
\[ \tau = it \]

\[ \hbar = h/(2\pi) \] Planck’s constant
\[ [h] = ML^2T^{-1} \] physical (engineering) dimension of \( h \)
\[ \nu \] frequency, \( \omega \) pulsation
\[ (1.70) \]
\[ (1.100) \]

Superanalysis
(Chapter 9)
\[ \tilde{A} \] parity of \( A \in \{0, 1\} \)
\[ AB = (-1)^{\tilde{A} \tilde{B}} BA \] graded commutativity
\[ [A, B] \] graded commutator (9.5)
\[ \{A, B\} \] graded anticommutator (9.6)
\[ \wedge = (-1)^{\tilde{A} \tilde{B}} B \wedge A \] graded exterior algebra
\[ \xi^\mu \xi^\sigma = -\xi^\sigma \xi^\mu \] Grassmann generators (9.11)
\[ z = u + v \] supernumber, \( u \) even \( \in \mathbb{C}_c \), \( v \) odd \( \in \mathbb{C}_a \) (9.12)
\[ \mathbb{R}_c \subset \mathbb{C}_c \] real elements of \( \mathbb{C}_c \) (9.16)
\[ \mathbb{R}_a \subset \mathbb{C}_a \] real elements of \( \mathbb{C}_a \) (9.16)
\[ z = z_B + z_S \] supernumber; \( z_B \) body, \( z_S \) soul (9.12)
\[ x^A = (x^a, \xi^a) \in \mathbb{R}^{n+1}_\nu \] superpoints (9.17)
\[ z = c_0 + c_i \xi^i + \frac{1}{2!} c_{ij} \xi^i \xi^j + \cdots \]
\[ = \rho + i\sigma, \]
\[ z^* := \rho - i\sigma, \]
\[ z^\dagger = \rho - i\sigma, \] where both \( \rho \) and \( \sigma \) have real coefficients
\[ (zz')^* = z^* z'^* \] (9.13)
\[ \text{hermitian conjugate,} \ (zz')^\dagger = z'^\dagger z^\dagger \]

Conventions
We use traditional conventions unless there is a compelling reason for using a different one. If a sign is hard to remember, we recall its origin.

Metric signature on pseudoriemannian spaces
\[ \eta_{\mu\nu} = \text{diag}(+,-,-,-) \]
\[ p_\mu p^\mu = (p^0)^2 - |\vec{p}|^2 = m^2 c^2, \] \( p^0 = E/c \)
\[ p_\mu x^\mu = Et - \vec{p} \cdot \vec{x}, \] \( x^0 = ct \)
\[ E = h\omega = h\nu, \] \( \vec{p} = h \vec{k} \), plane wave \( \omega = \vec{v} \cdot \vec{k} \)

Positive-energy plane wave \( \exp(-ip_\mu x^\mu/h) \)
Clifford algebra
\[ \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \]
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Quantum operators

\[ [p_\mu, x^\nu] = -i\hbar\delta^\nu_\mu \Rightarrow p_\mu = -i\hbar\partial_\mu \]

Quantum physics (time \( t \)) and statistical mechanics (parameter \( \tau \))

\( \tau = it \) (see (1.100))

Physical dimension

\[ [\hbar] = ML^2T^{-1} \]

\[ h^{-1}\langle p, x \rangle = \frac{2\pi}{\hbar}\langle p, x \rangle \text{ is dimensionless} \]

Fourier transforms

\[ (\mathcal{F} f)(x') := \int_{\mathbb{R}^D} d^Dx \exp(-2\pi i\langle x', x \rangle)f(x) \quad x \in \mathbb{R}^D, x' \in \mathbb{R}^D \]

For Grassmann variables

\[ (\mathcal{F} f)(\kappa) := \int \delta \xi \exp(-2\pi i\kappa \xi)f(\xi) \]

In both cases

\[ \langle \delta, f \rangle = f(0) \quad \text{i.e. } \delta(\xi) = C^{-1}\xi \]

\[ \mathcal{F} \delta = 1 \quad \text{i.e. } C^2 = (2\pi i)^{-1} \]

\[ \int \delta \xi \xi = C, \quad \text{here } C^2 = (2\pi i)^{-1} \]

**Formulary** (giving a context to symbols)

- Wiener integral

\[ \mathbb{E}\left[ \exp\left( -\int_{\tau_a}^{\tau_b} d\tau V(q(\tau)) \right) \right] \quad (1.1) \]

- Peierls bracket

\[ (A, B) := \mathcal{D}_A^\dagger B - (-1)^{\bar{A}\bar{B}}\mathcal{D}_B^\dagger A \quad (1.9) \]

- Schwinger variational principle

\[ \delta\langle A|B \rangle = i\langle A|\delta S/\hbar|B \rangle \quad (1.11) \]

- Quantum partition function

\[ Z(\beta) = \text{Tr}(e^{-\beta H}) \quad (1.71) \]
List of symbols, conventions and formulary

- Schrödinger equation
  \[
  \begin{aligned}
  i\hbar \partial_t \psi(x,t) &= (-\frac{1}{2} \mu^2 \Delta_x + \hbar^{-1} V(x)) \psi(x,t) \\
  \psi(x,t_a) &= \phi(x)
  \end{aligned}
  \]  
  (1.77)

  \[\mu^2 = \hbar/m\]

- Gaussian integral
  \[
  \int_X d\Gamma_{s,Q}(x) \exp(-2\pi i \langle x',x \rangle) := \exp(-s\pi W(x'))
  \]  
  (2.29)\textsubscript{s}

  \[d\Gamma_{s,Q}x = \mathcal{D}_{s,Q}(x) \exp\left(-\frac{\pi}{s} Q(x)\right)\]
  (2.30)\textsubscript{s}

  \[
  Q(x) = \langle Dx, x \rangle, \quad W(x') = \langle x', Gx' \rangle
  \]  
  (2.28)

- Sum without repetition
  \[
  \int_X d\Gamma_{s,Q}(x) \langle x_1', x \rangle \cdots \langle x_{2n}', x \rangle = \left(\frac{s}{2\pi}\right)^n \sum' W(x_1', x_{i_2}) \cdots W(x_{i_{2n-1}}', x_{i_{2n}})
  \]

- Linear maps
  \[
  \langle \tilde{L} y', x \rangle = \langle y', Lx \rangle
  \]  
  (2.58)

  \[W_{\tilde{Y}} = W_{\tilde{X}} \circ \tilde{L}, \quad Q_{\tilde{X}} = Q_Y \circ L\]
  (Chapter 3, box)

- Scaling and coarse graining (Section 2.5)
  \[
  S_l u(x) = l^{[\nu]} u \left( \frac{x}{l} \right)
  \]

  \[S_l[a,b] = \begin{bmatrix} a & b \\ \overline{T} & \overline{l} \end{bmatrix}, \quad P_l := S_{l/l_0} \cdot \mu_{[l_0,l]}^*[\nu]\]
  (2.94)

- Jacobi operator
  \[
  S''(q) \cdot \xi \xi = \langle \mathcal{J}(q) \cdot \xi, \xi \rangle
  \]  
  (5.7)

- Operator formalism
  \[
  \langle b|\hat{O}|a\rangle = \int_{P_{a,b}} O(\gamma) \exp(iS(\gamma)/\hbar) \mu(\gamma) \mathcal{D}\gamma
  \]  
  (Chapter 6, box)

- Time-ordered exponential
  \[
  T \exp \left( \int_{t_0}^t ds A(s) \right)
  \]  
  (6.38)
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- Dynamical vector fields

\[ dx(t, z) = X_A(x(t, z))dz^A(t) + Y(x(t, z))dt \] (7.14)

\[ \Psi(t, x_0) := \int_{\mathbb{R}^D} Ds Q_0 \exp \left( -\frac{\pi}{s} Q_0(z) \right) \phi(x_0 \cdot \Sigma(t, z)) \] (7.12)

\[ Q_0(z) := \int_T dt h_{AB} \dot{z}^A(t) \dot{z}^B(t) \] (7.8)

\[ \left\{ \begin{array}{l}
\partial \Psi \over \partial t = \frac{s}{4\pi} h_{AB} L_{X_A} L_{X_B} \Psi + L_Y \Psi \\
\Psi(t_0, x) = \phi(x)
\end{array} \right. \] (7.15)

- Homotopy

\[ |K(b, t_b; a, t_a)| = \left| \sum_{\alpha} \chi(\alpha) K^\alpha(b, t_b; a, t_a) \right| \] (Chapter 8, box)

- Koszul formula

\[ L_X \omega = \text{Div}_\omega(X) \cdot \omega \] (11.1)

- Miscellaneous

\begin{align*}
\text{det exp} \ A &= \exp \text{tr} \ A \\
d \ln \text{det} \ A &= \text{tr}(A^{-1} \, dA) \\
\nabla^i f &\equiv \nabla^g \cdots f := g^{ij} \partial f / \partial x^j \quad \text{gradient} \\
(\nabla g^{-1} | V)_g &= V^j, j \quad \text{divergence} \\
(V | \nabla f) &= - (\text{div} V | f) \quad \text{gradient/divergence}
\end{align*} (11.47 - 11.79)

- Poisson processes

\[ N(t) := \sum_{k=1}^{\infty} \theta(t - T_k) \quad \text{counting process} \] (13.17)

- Density of energy states

\[ \rho(E) = \sum_n \delta(E - E_n), \quad H \psi_n = E_n \psi_n \]

- Time ordering

\[ T(\phi(x_j)\phi(x_i)) = \begin{cases} 
\phi(x_j)\phi(x_i) & \text{for } j > i \\
\phi(x_i)\phi(x_j) & \text{for } i > j
\end{cases} \] (15.7)
List of symbols, conventions and formulary

- The “measure” (Chapter 18)
  \[ \mu[\phi] \approx (\text{sdet } G^+[\phi])^{-1/2} \]  \hspace{1cm} (18.3)
  \[ i_i S_{k[j]} G^{+ki} [\phi] = - i \delta^i_j \]  \hspace{1cm} (18.4)
  \[ G^{+ij}[\phi] = 0 \quad \text{when } i \gg j \]  \hspace{1cm} (18.5)
  \[ \phi^i = \begin{cases} u^i_{\text{in } A} a^A_{\text{in}} + u^i_{\text{in } A} a^A_{\text{in}}^* \\ u^i_{\text{out } X} a^X_{\text{out}} + u^i_{\text{out } X} a^X_{\text{out}}^* \end{cases} \]  \hspace{1cm} (18.18)

- Wick (normal ordering)
  operator normal ordering
  \[ (a + a^\dagger)(a + a^\dagger) = (a + a^\dagger)^2 : 1 \]  \hspace{1cm} (D.1)
  functional normal ordering
  \[ :F(\phi) :G := \exp \left( -\frac{1}{2} \Delta_G \right) F(\phi) \]  \hspace{1cm} (D.4)
  functional laplacian defined by the covariance \( G \)
  \[ \Delta_G := \int_{M^D} d^D x \int_{M^D} d^D y \ G(x, y) \frac{\delta^2}{\delta \phi(x) \delta \phi(y)} \]  \hspace{1cm} (2.63)