1 Numbers

In this book we study the properties of real functions defined on intervals of the real line (possibly the whole real line) and whose image also lies on the real line. In other words, they map $\mathbb{R}$ into $\mathbb{R}$. Our work will be from a very precise point of view in order to establish many of the properties of such functions which seem intuitively obvious; in the process we will discover that some apparently true properties are in fact not necessarily true!

The types of functions that we shall examine include:

exponential functions, such as $x \mapsto a^x$, where $a, x \in \mathbb{R}$,
trigonometric functions, such as $x \mapsto \sin x$, where $x \in \mathbb{R}$,
root functions, such as $x \mapsto \sqrt{x}$, where $x \geq 0$.

The types of behaviour that we shall examine include *continuity*, *differentiability* and *integrability* – and we shall discover that functions with these properties can be used in a number of surprising applications.

However, to put our study of such functions on a secure foundation, we need first to clarify our ideas of the *real numbers* themselves and their properties. In particular, we need to devote some time to the manipulation of *inequalities*, which play a key role throughout the book.

For example, what exactly *is* the number $\sqrt{2}$?

In Section 1.1, we start by revising the properties of rational numbers and their *decimal representation*. Then we introduce the real numbers as infinite decimals, and describe the difficulties involved in doing arithmetic with such decimals.

In Section 1.2, we revise the rules for manipulating inequalities and show how to find the *solution set* of an inequality involving a real number, x, by applying the rules. We also explain how to deal with inequalities which involve *modulus* signs.

In Section 1.3, we describe various techniques for proving inequalities, including the very important technique of *Mathematical Induction*.

The concept of a *least upper bound*, which is of great importance in Analysis, is introduced in Section 1.4, and we discuss the *Least Upper Bound Property* of $\mathbb{R}$.

Finally, in Section 1.5, we describe how least upper bounds can be used to define arithmetical operations in $\mathbb{R}$.

You may omit this section at a first reading.

Even though you may be familiar with much of this material we recommend that you read through it, as we give the system of real numbers a more careful treatment than you may have met before. The material on inequalities and least upper bounds is particularly important for later on.

In later chapters we shall define exactly what the numbers π and e are, and find various ways of calculating them. But, first, we examine numbers in general.

1.1 Real numbers

We start our study of the real numbers with the rational numbers, and investigate their decimal representations, then we proceed to the irrational numbers.

1.1.1 Rational numbers

We assume that you are familiar with the set of **natural numbers**

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

Note that 0 is *not* a natural number.

and with the set of **integers**

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The set of **rational numbers** consists of all fractions (or ratios of integers)

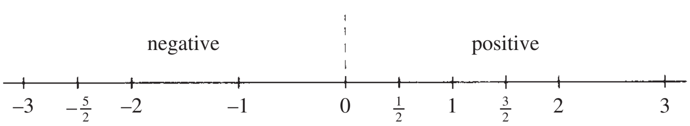
$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\}.$$

Remember that each rational number has many different representations as a ratio of integers; for example

$$\frac{1}{3} = \frac{2}{6} = \frac{10}{30} = \ldots.$$

We also assume that you are familiar with the usual arithmetical operations of addition, subtraction, multiplication and division of rational numbers.

It is often convenient to represent rational numbers geometrically as points on a **number line**. We begin by drawing a line and marking on it points corresponding to the integers 0 and 1. If the distance between 0 and 1 is taken as a unit of length, then the rationals can be arranged on the line with positive rationals to the right of 0 and negative rationals to the left.



For example, the rational $\frac{3}{2}$ is placed at the point which is one-half of the way from 0 to 3.

This means that rationals have a natural order on the number-line. For example, $\frac{19}{22}$ lies to the left of $\frac{7}{8}$ because

$$\frac{19}{22} = \frac{76}{88} \quad \text{and} \quad \frac{7}{8} = \frac{77}{88}.$$

If a lies to the left of b on the number-line, then

$$a \text{ is } \textit{less than } b \quad \text{or} \quad b \text{ is } \textit{greater than } a,$$

and we write

$$a < b \quad \text{or} \quad b > a.$$

For example

$$\frac{19}{22} < \frac{7}{8} \quad \text{or} \quad \frac{7}{8} > \frac{19}{22}.$$

We write $a \le b$, or $b \ge a$, if either $a < b$ or $a = b$.

Problem 1 Arrange the following rational numbers in order:

$$0, 1, -1, \tfrac{17}{20}, -\tfrac{17}{20}, \tfrac{45}{53}, -\tfrac{45}{53}.$$

Problem 2 Show that between any two distinct rational numbers there is another rational number.

1.1.2 Decimal representation of rational numbers

The decimal system enables us to represent all the natural numbers using only the ten integers

$$0, 1, 2, 3, 4, 5, 6, 7, 8 \text{ and } 9,$$

which are called *digits*. We now remind you of the basic facts about the representation of *rational* numbers by decimals.

Definition A **decimal** is an expression of the form

$$\pm a_0 \cdot a_1 a_2 a_3 \ldots,$$

where a_0 is a non-negative integer and $a_1, a_2, a_3, \ldots$ are digits.

For example
$0.8500\ldots,$
$13.1212\ldots,$
$-1.111\ldots.$

If only a finite number of the digits $a_1, a_2, a_3, \ldots$ are non-zero, then the decimal is called **terminating** or **finite**, and we usually omit the tail of 0s.

For example, $0.8500\ldots = 0.85.$

Terminating decimals are used to represent rational numbers in the following way

$$\pm a_0 \cdot a_1 a_2 a_3 \ldots a_n = \pm\left(a_0 + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}\right).$$

For example
$$0.85 = 0 + \frac{8}{10^1} + \frac{5}{10^2} = \frac{85}{100} = \frac{17}{20}.$$

It can be shown that any fraction whose denominator contains only powers of 2 and/or 5 (such as $20 = 2^2 \times 5$) can be represented by such a terminating decimal, which can be found by long division.

However, if we apply long division to many other rationals, then the process of long division never terminates and we obtain a **non-terminating** or **infinite** decimal. For example, applying long division to $\frac{1}{3}$ gives $0.333\ldots$, and for $\frac{19}{22}$ we obtain $0.86363\ldots$.

Problem 3 Apply long division to $\frac{1}{7}$ and $\frac{2}{13}$ to find the corresponding decimals.

These non-terminating decimals, which are obtained by applying the long division process, have a certain common property. All of them are **recurring**; that is, they have a recurring block of digits, and so can be written in shorthand form, as follows:

$$\begin{aligned} 0.333\ldots &= 0.\overline{3}, \\ 0.142857142857\ldots &= 0.\overline{142857}\ldots, \\ 0.86363\ldots &= 0.8\overline{63}. \end{aligned}$$

Another commonly used notation is
$0.\dot{3}$ or $0.\dot{1}4285\dot{7}$.

It is not hard to show, whenever we apply the long division process to a fraction $\frac{p}{q}$, that the resulting decimal is recurring. To see why we notice that there are only q possible remainders at each stage of the division, and so one of these remainders must eventually recur. If the remainder 0 occurs, then the resulting decimal is, of course, terminating; that is, it ends in recurring 0s.

Non-terminating recurring decimals which arise from the long division of fractions are used to represent the corresponding rational numbers. This representation is not quite so straight-forward as for terminating decimals, however. For example, the statement

$$\frac{1}{3} = 0.\overline{3} = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$$

can be made precise only when we have introduced the idea of the *sum of a convergent infinite series*. For the moment, when we write the statement $\frac{1}{3} = 0.\overline{3}$ we mean simply that the decimal $0.\overline{3}$ arises from $\frac{1}{3}$ by the long division process.

We return to this topic in Chapter 3.

The following example illustrates one way of finding the rational number with a given decimal representation.

Example 1 Find the rational number (expressed as a fraction) whose decimal representation is $0.8\overline{63}$.

Solution First we find the fraction x such that $x = 0.\overline{63}$.

If we multiply both sides of this equation by 10^2 (because the recurring block has length 2), then we obtain

$$100x = 63.\overline{63} = 63 + x.$$

Hence

$$99x = 63 \Rightarrow x = \frac{63}{99} = \frac{7}{11}.$$

Thus

$$0.8\overline{63} = \frac{8}{10} + \frac{x}{10} = \frac{8}{10} + \frac{7}{110} = \frac{95}{110} = \frac{19}{22}. \qquad \square$$

The key idea in the above solution is that multiplication of a decimal by 10^k moves the decimal point k places to the right.

Problem 4 Using the above method, find the fractions whose decimal representations are:

(a) $0.\overline{231}$; (b) $2.2\overline{81}$.

The decimal representation of rational numbers has the advantage that it enables us to decide immediately which of two distinct positive rational numbers is the greater. We need only examine their decimal representations and notice the first place at which the digits differ. For example, to order $\frac{7}{8}$ and $\frac{19}{22}$ we write

$$\frac{7}{8} = 0.875\ldots \quad \text{and} \quad \frac{19}{22} = 0.86363\ldots,$$

and so

$$0.8\overset{\downarrow}{6}363\ldots < 0.8\overset{\downarrow}{7}5 \Rightarrow \frac{19}{22} < \frac{7}{8}.$$

Problem 5 Find the first two digits after the decimal point in the decimal representations of $\frac{17}{20}$ and $\frac{45}{53}$, and hence determine which of these two rational numbers is the greater.

Warning Decimals which end in recurring 9s sometimes arise as alternative representations for terminating decimals. For example

$$1 = 0.\overline{9} = 0.999\ldots \quad \text{and} \quad 1.35 = 1.34\overline{9} = 1.34999\ldots.$$

Whenever possible, we avoid using the form of a decimal which ends in recurring 9s.

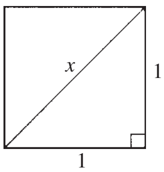
You may find this rather alarming, but it is important to realise that this is a matter of *convention*. We wish to allow the decimal 0.999... to represent a number x, so x must be less than or equal to 1 and greater than each of the numbers

$$0.9,\ 0.99,\ 0.999,\ \ldots.$$

The *only* rational with these properties is 1.

1.1.3 Irrational numbers

One of the surprising mathematical discoveries made by the Ancient Greeks was that the system of rational numbers is not adequate to describe all the magnitudes that occur in geometry. For example, consider the diagonal of a square of side 1. What is its length? If the length is x, then, by Pythagoras' Theorem, x must satisfy the equation $x^2 = 2$. However, there is no rational number which satisfies this equation.



$x^2 = 1^2 + 1^2 = 2$

Theorem 1 There is no rational number x such that $x^2 = 2$.

Proof Suppose that such a rational number x exists. Then we can write $x = \frac{p}{q}$. By cancelling, if necessary, we may assume that p and q have no common factor. The equation $x^2 = 2$ now becomes

This is a proof by contradiction.

$$\frac{p^2}{q^2} = 2, \quad \text{so} \quad p^2 = 2q^2.$$

Now, the square of an odd number is odd, and so p cannot be odd. Hence p is even, and so we can write $p = 2r$, say. Our equation now becomes

For

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1.$$

$$(2r)^2 = 2q^2, \quad \text{so} \quad q^2 = 2r^2.$$

Reasoning as before, we find that q is also even.

Since p and q are both even, they have a common factor 2, which contradicts our earlier statement that p and q have no common factors.

Arguing from our original assumption that x exists, we have obtained two contradictory statements. Thus, our original assumption must be false. In other words, no such x exists. ■

Problem 6 By imitating the above proof, show that there is no rational number x such that $x^3 = 2$.

Since we want equations such as $x^2 = 2$ and $x^3 = 2$ to have solutions, we must introduce new numbers which are not rational numbers. We denote these new numbers by $\sqrt{2}$ and $\sqrt[3]{2}$, respectively; thus $\left(\sqrt{2}\right)^2 = 2$ and $\left(\sqrt[3]{2}\right)^3 = 2$. Of course, we must introduce many other new numbers, such as $\sqrt{3}, \sqrt[5]{11}$, and so on. Indeed, it can be shown that, if m, n are natural numbers and $x^m = n$ has no integer solution, then $\sqrt[m]{n}$ cannot be rational. A number which is not rational is called **irrational**.

The case $m = 2$ is treated in Exercise 5 for this section in Section 1.6.

There are many other mathematical quantities which cannot be described exactly by rational numbers. For example, the number π which denotes the area of a disc of radius 1 (or half the length of the perimeter of such a disc) is irrational, as is the number e.

Lambert proved that π is irrational in 1770.

It is natural to ask whether irrational numbers, such as $\sqrt{2}$ and π, can be represented as decimals. Using your calculator, you can check that $(1.41421356)^2$ is very close to 2, and so 1.41421356 is a very good approximate value for $\sqrt{2}$. But is there a decimal which represents $\sqrt{2}$ *exactly*? If such a decimal exists, then it cannot be recurring, because all the recurring decimals correspond to rational numbers.

In fact $(1.41421356)^2 = 1.9999999932878736.$

In fact, it *is* possible to represent all the irrational numbers mentioned so far by non-recurring decimals. For example, there are non-recurring decimals such that

$$\sqrt{2} = 1.41421356\ldots \quad \text{and} \quad \pi = 3.14159265\ldots.$$

We prove that $\sqrt{2}$ has a decimal representation in Section 1.5.

It is also natural to ask whether non-recurring decimals, such as

$$0.101001000100001\ldots \quad \text{and} \quad 0.123456789101112\ldots,$$

represent irrational numbers. In fact, a decimal corresponds to a rational number if and only if it is recurring; so a non-recurring decimal must correspond to an irrational number.

We may summarise this as:

recurring decimal $\Leftrightarrow$ *rational number*

non-recurring decimal $\Leftrightarrow$ *irrational number*

1.1.4 The real number system

Taken together, the rational numbers (recurring decimals) and irrational numbers (non-recurring decimals) form the set of **real numbers**, denoted by $\mathbb{R}$.

As with rational numbers, we can determine which of two real numbers is greater by comparing their decimals and noticing the first pair of corresponding digits which differ. For example

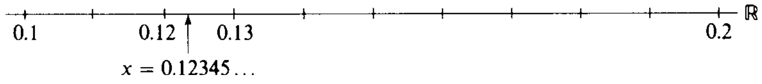
$$0.1\overset{\downarrow}{0}100100010000\ldots < 0.1\overset{\downarrow}{2}3456789101112\ \ldots.$$

When comparing decimals in this way, we do not allow either decimal to end in recurring 9s.

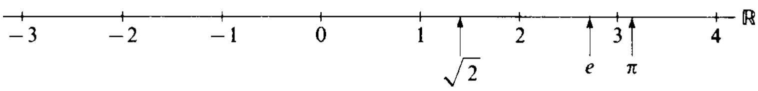
We now associate with each irrational number a point on the number-line. For example, the irrational number $x = 0.123456789101112\ldots$ satisfies each of the inequalities

$$0.1 < x < 0.2$$
$$0.12 < x < 0.13$$
$$0.123 < x < 0.124$$
$$\vdots$$

We assume that there is a point on the number-line corresponding to x, which lies to the right of each of the (rational) numbers 0.1, 0.12, 0.123 …, and to the left of each of the (rational) numbers 0.2, 0.13, 0.124, ….



As usual, negative real numbers correspond to points lying to the left of 0; and the 'number-line', complete with both rational and irrational points, is called the **real line**.



There is thus a one–one correspondence between the points on the real line and the set $\mathbb{R}$ of real numbers (or decimals).

We now state several properties of $\mathbb{R}$, with which you will be already familiar, although you may not have met their names before. These properties are used frequently in Analysis, and we do not always refer to them explicitly by name.

Order Properties of $\mathbb{R}$

1. **Trichotomy Property** If $a, b \in \mathbb{R}$, then *exactly one* of the following inequalities holds
$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$
2. **Transitive Property** If $a, b, c \in \mathbb{R}$, then
$$a < b \quad \text{and} \quad b < c \Rightarrow a < c.$$
3. **Archimedean Property** If $a \in \mathbb{R}$, then there is a positive integer n such that
$$n > a.$$
4. **Density Property** If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational number x and an irrational number y such that
$$a < x < b \quad \text{and} \quad a < y < b.$$

The first three of these properties are almost self-evident, but the Density Property is not so obvious.

Remark

The Archimedean Property is sometimes expressed in the following equivalent way: for any positive real number a, there is a positive integer n such that $\frac{1}{n} < a$.

The following example illustrates how we can prove the Density Property.

Example 2 Find a rational number x and an irrational number y satisfying
$$a < x < b \quad \text{and} \quad a < y < b,$$
where $a = 0.12\overline{3}$ and $b = 0.12345\ldots$.

Solution The two decimals
$$a = 0.123\overset{\downarrow}{3}\ldots \quad \text{and} \quad b = 0.123\overset{\downarrow}{4}5\ldots$$
differ first at the fourth digit. If we truncate b after this digit, we obtain the rational number $x = 0.1234$, which satisfies the requirement that $a < x < b$.

To find an irrational number y between a and b, we attach to x a (sufficiently small) non-recurring tail such as $010010001\ldots$ to give $y = 0.1234|010010001\ldots$. It is then clear that y is irrational (because its decimal is non-recurring) and that $a < y < b$. □

Problem 7 Find a rational number x and an irrational number y such that $a < x < b$ and $a < y < b$, where $a = 0.\overline{3}$ and $b = 0.3401$.

Theorem 2 Density Property of $\mathbb{R}$
If a, $b \in \mathbb{R}$ and $a < b$, then there is a rational number x and an irrational number y such that

$$a < x < b \quad \text{and} \quad a < y < b.$$

You may omit the following proof on a first reading.

Proof For simplicity, we assume that $a, b \geq 0$. So, let a and b have decimal representations

$$a = a_0 \cdot a_1 a_2 a_3 \ldots \quad \text{and} \quad b = b_0 \cdot b_1 b_2 b_3 \ldots,$$

Here a_0, b_0 are non-negative integers, and $a_1, b_1, a_2, b_2, \ldots$ are digits.

where we arrange that a does not end in recurring 9s, whereas b does not terminate (this latter can be arranged by replacing a terminating representation by an equivalent representation that ends in recurring 9s).

Since $a < b$, there must be some integer n such that

$$a_0 = b_0,\ a_1 = b_1,\ \ldots,\ a_{n-1} = b_{n-1},\ \text{but } a_n < b_n.$$

Then $x = a_0 \cdot a_1 a_2 a_3 \ldots a_{n-1} b_n$ is rational, and $a < x < b$ as required.

Finally, since $x < b$, it follows that we can attach a sufficiently small non-recurring tail to x to obtain an irrational number y for which $a < y < b$. ■

Remark

One consequence of the Density Property is that between any two real numbers there are infinitely many rational numbers and infinitely many irrational numbers.

Problem 8 Prove that between any two real numbers a and b there are at least two distinct rational numbers.

A proof of the previous remark would involve ideas similar to those involved in tackling Problem 8.

1.1.5 Arithmetic in $\mathbb{R}$

We can do arithmetic with *recurring* decimals by first converting the decimals to fractions. However, it is not obvious how to perform arithmetical operations with *non-recurring* decimals.

Assuming that we can represent $\sqrt{2}$ and π by the non-recurring decimals

$$\sqrt{2} = 1.41421356\ldots \quad \text{and} \quad \pi = 3.14159265\ldots,$$

can we also represent the sum $\sqrt{2} + \pi$ and the product $\sqrt{2} \times \pi$ as decimals? Indeed, what do we mean by the operations of addition and multiplication when non-recurring decimals (irrationals) are involved, and do these operations satisfy the same properties as addition and multiplication of rationals?

It would take many pages to answer these questions fully. Therefore, we shall *assume* that it is possible to define all the usual arithmetical operations with decimals, and that they do satisfy the usual properties. For definiteness, we now list these properties.

Arithmetic in $\mathbb{R}$

	Addition		*Multiplication*	
A1	If $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$.	M1	If $a, b \in \mathbb{R}$, then $a \times b \in \mathbb{R}$.	CLOSURE
A2	If $a \in \mathbb{R}$, then $a + 0 = 0 + a = a$.	M2	If $a \in \mathbb{R}$, then $a \times 1 = 1 \times a = a$.	IDENTITY
A3	If $a \in \mathbb{R}$, then there is a number $-a \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.	M3	If $a \in \mathbb{R} - \{0\}$, then there is a number $a^{-1} \in \mathbb{R}$ such that $a \times a^{-1} = a^{-1} \times a = 1$.	INVERSES
A4	If $a, b, c \in \mathbb{R}$, then $(a + b) + c = a + (b + c)$.	M4	If $a, b, c \in \mathbb{R}$, then $(a \times b) \times c = a \times (b \times c)$.	ASSOCIATIVITY
A5	If $a, b \in \mathbb{R}$, then $a + b = b + a$.	M5	If $a, b \in \mathbb{R}$, then $a \times b = b \times a$.	COMMUTATIVITY
D	If $a, b, c \in \mathbb{R}$, then $a \times (b + c) = a \times b + a \times c$.			DISTRIBUTATIVITY

To summarise the contents of this table:

- $\mathbb{R}$ is an Abelian group under the operation of addition $+$; (Properties A1–A5)
- $\mathbb{R} - \{0\}$ is an Abelian group under the operation of multiplication $\times$; (Properties M1–M5)
- These two group structures are linked by the Distributive Property. (Property D)

It follows from the above properties that we can perform addition, subtraction (where $a - b = a + (-b)$), multiplication and division (where $\frac{a}{b} = a \times b^{-1}$) in $\mathbb{R}$, and that these operations satisfy all the usual properties.

Any system satisfying the properties listed in the table is called a **field**. Both $\mathbb{Q}$ and $\mathbb{R}$ are fields.

Furthermore, we shall assume that the set $\mathbb{R}$ contains the nth roots and rational powers of positive real numbers, with their usual properties. In Section 1.5 we describe one way of justifying the existence of nth roots.

1.2 Inequalities

Much of Analysis is concerned with inequalities of various kinds; the aim of this section and the next section is to provide practice in the manipulation of inequalities.

1.2.1 Rearranging inequalities

The fundamental rule, upon which much manipulation of inequalities is based, is that the statement $a < b$ means exactly the same as the statement $b - a > 0$. This fact can be stated concisely in the following way:

Recall that the symbol '$\Leftrightarrow$' means 'if and only if' or 'implies and is implied by'.

Rule 1 For any $a, b \in \mathbb{R}$, $\quad a < b \Leftrightarrow b - a > 0$.

Put another way, the inequalities $a < b$ and $b - a > 0$ are *equivalent*.

There are several other standard rules for rearranging a given inequality into an equivalent form. Each of these can be deduced from our first rule above. For example, we obtain an equivalent inequality by adding the same expression to both sides.

Rule 2 For any $a, b, c \in \mathbb{R}$, $a < b \Leftrightarrow a + c < b + c$.

Another way to rearrange an inequality is to multiply both sides by a non-zero expression, making sure to *reverse* the inequality if the expression is negative.

Rule 3
- For any $a, b \in \mathbb{R}$ and any $c > 0$, $a < b \Leftrightarrow ac < bc$;
- For any $a, b \in \mathbb{R}$ and any $c < 0$, $a < b \Leftrightarrow ac > bc$.

For example

$2 < 3 \Leftrightarrow 20 < 30$ $(c = 10)$,

$2 < 3 \Leftrightarrow -20 > -30$ $(c = -10)$.

Sometimes the most effective way to rearrange an inequality is to take reciprocals. However, in this case, both sides of the inequality should be positive, and the direction of the inequality has to be *reversed*.

Rule 4 (Reciprocal Rule)
For any positive $a, b \in \mathbb{R}$, $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$.

For example

$2 < 4 \Leftrightarrow \frac{1}{2} (=0.5) > \frac{1}{4} (=0.25)$.

Some inequalities can be simplified only by taking powers. However, in order to do this, both sides must be non-negative and must be raised to a *positive* power.

Rule 5 (Power Rule)
For any non-negative $a, b \in \mathbb{R}$, and any $p > 0$, $a < b \Leftrightarrow a^p < b^p$.

For example

$4 < 9 \Leftrightarrow 4^{\frac{1}{2}} (=2) < 9^{\frac{1}{2}} (=3)$.

We shall discuss the meaning of non-integer powers in Section 1.5.

For positive integers p, Rule 5 follows from the identity

$$b^p - a^p = (b - a)\left(b^{p-1} + b^{p-2}a + \cdots + ba^{p-2} + a^{p-1}\right);$$

thus, since the right-hand bracket is positive, we have

$$b - a > 0 \Leftrightarrow b^p - a^p > 0,$$

which is equivalent to our desired result.

For example,
$b^3 - a^3 = (b - a) \times (b^2 + ba + a^2)$.

Remark

There are corresponding versions of Rules 1–5 in which the *strict* inequality $a < b$ is replaced by the *weak* inequality $a \leq b$.

Problem 1 State (without proof) the versions of Rules 1–5 for weak inequalities.

We shall give one more rule for rearranging inequalities in Sub-section 1.2.3.

1.2.2 Solving inequalities

Solving an inequality involving an unknown real number x means determining those values of x for which the given inequality holds; that is, finding the *solution set* of the inequality. We can often do this by rewriting the inequality in an equivalent, but simpler form, using the rules given in the last sub-section.

The *solution set* is the set of those values of x for which the inequality holds.