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Roger J. Gray and Susan M. Pitts
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Introduction

1.1 The aim of this book

Knowledge of risk models and the assessment of risk will be of great importance to actuaries as they apply their skills and expertise today and in the future. The title of this book "Risk Modelling in General Insurance: From Principles to Practice" reflects our intention to present a wide range of statistical and probabilistic topics relevant to actuarial methodology in general insurance. Our aim is to achieve this in a focused and coherent manner, which will appeal to actuarial students and others interested in the topics we cover.

We believe that the material is suitable for advanced undergraduates and students taking master's degree courses in actuarial science, and also those taking mathematics and statistics courses with some insurance mathematics content. In addition, students with a strong quantitative/mathematical background taking economics and business courses should also find much of interest in the book. Prerequisites for readers to benefit fully from the book include first undergraduate-level courses in calculus, probability and statistics. We do not assume measure theory.

Our aim is that readers who master the content will extend their knowledge effectively and will build a firm foundation in the statistical and actuarial concepts and their applications covered. We hope that the approach and content will engage readers and encourage them to develop and extend their critical and comparative skills. In particular, our aim has been to provide opportunities for readers to improve their higher-order skills of analysis and synthesis of ideas across topics.

A key feature of our approach is the inclusion of a large number of worked examples and extensive sets of exercises, which we think readers will find stimulating. In addition, we include three case studies, each of which brings

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together a number of concepts and applications from different parts of the book.

While the book covers much of the international syllabuses for professional actuarial examinations in risk models, it goes further and deeper in places.

The book includes appropriate references to the open source (free and easily downloadable) statistical software package ***R*** throughout, giving readers opportunities to learn how simple code and functions can be used profitably in an actuarial context.

1.2 Notation and prerequisites

The tools of probability theory are crucial for the study of the risk models in this book, and, in §1.2.1, we give an overview of the required basic concepts of probability. This overview also serves to introduce the notation that we will use throughout the book. In §1.2.2 and §1.2.3, we indicate the assumed prerequisites in statistics and simulation, and finally in §1.2.4 we give information about the statistical software package ***R***.

1.2.1 Probability

We start with definitions and notation for basic quantities related to a random variable X. Our first such quantity is the distribution function (or *cumulative distribution function*) F_X of X, given by

$$F_X(x) = \Pr(X \leq x), \quad x \in \mathbb{R}.$$

The function F_X is non-decreasing and right-continuous. It satisfies $0 \leq F_X(x) \leq 1$ for all x in $\mathbb{R}$, $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$. Most of the random variables in this book are non-negative, i.e. they take values in $[0, \infty)$. If V is a non-negative random variable, then we assume without comment that $F_V(v) = 0$ for $v < 0$. For a non-negative random variable V, the *tail* of F_V is $\Pr(V > v) = 1 - F_V(v)$ for $v \geq 0$.

A *continuous* random variable Y has a *probability density function* f_Y, which is a non-negative function f_Y, with $\int_{-\infty}^{\infty} f_Y(y)dy = 1$, such that the distribution function of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt, \quad y \in \mathbb{R}.$$

This means that F_Y is a continuous function. The probability that Y is in a set A is $\Pr(Y \in A) = \int_A f_Y(y)dy$. (For those readers who are familiar with measure

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theory, note that we will tacitly assume the word "measurable" where necessary. Those readers who are not familiar with measure theory may ignore this remark, but may like to note that a rigorous treatment of probability theory requires more careful definitions and statements than appear in introductory courses and in this overview.)

Let N be a *discrete* random variable that takes values in $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then $\Pr(N = x)$, $x \in \mathbb{R}$, is the *probability mass function* of N. We see that $\Pr(N = x) = 0$ for $x \notin \mathbb{N}$, so that, for a discrete random variable concentrated on $\mathbb{N}$, the probability mass function is specified by $\Pr(N = k)$ for $k \in \mathbb{N}$. We then have $\sum_{k=0}^{\infty} \Pr(N = k) = 1$. The distribution function of N is

$$F_N(x) = \sum_{\{k:k\leq x\}} \Pr(N = k), \qquad x \in \mathbb{R},$$

and the graph of F_N is a non-decreasing step function, with an upward jump of size $\Pr(N = k)$ at k for all $k \in \mathbb{N}$. The probability that N is in a set A is

$$\Pr(N \in A) = \sum_{\{k:k\in A\}} \Pr(N = k).$$

We use the notation $\mathbb{E}[X]$ for the *expected value* (or *expectation*, or *mean*) of a random variable X. The expectation of the continuous random variable Y is

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy,$$

while for the discrete random variable N taking values in $\mathbb{N}$, the expectation is

$$\mathbb{E}[N] = \sum_{k=0}^{\infty} k \Pr(N = k).$$

We note that there are various possibilities for the expectation: it may be finite, it may take the value $+\infty$ or $-\infty$, or it may not be defined. The expectation of a non-negative random variable is either a finite non-negative value or $+\infty$.

For a real-valued function h on $\mathbb{R}$ and a continuous random variable Y, the expectation of $h(Y)$ is

$$\mathbb{E}[h(Y)] = \int_{-\infty}^{\infty} h(y) f_Y(y) dy,$$

whenever the integral is defined, and for a discrete random variable N taking values in $\mathbb{N}$, the expectation of $h(N)$ is

$$\mathbb{E}[h(N)] = \sum_{k=0}^{\infty} h(k) \Pr(N = k).$$

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For $r \geq 0$, the *rth moment* of X is $\mathbb{E}[X^r]$, when it is defined. The rth moment of a continuous random variable Y is

$$\int_{-\infty}^{\infty} y^r f_Y(y)dy,$$

and the rth moment of the discrete random variable N taking values in $\mathbb{N}$ is

$$\sum_{k=0}^{\infty} k^r \Pr(N = k).$$

Recall that if $\mathbb{E}[|X|^r]$ is finite for some $r > 0$, then $\mathbb{E}[|X|^s]$ is finite for all $0 \leq s \leq r$. Throughout the book, when we write down a particular moment such as $\mathbb{E}[N^3]$, then, unless otherwise stated, we assume that this moment is finite.

The *rth central moment* of a random variable X is $\mathbb{E}[(X-\mathbb{E}[X])^r]$. The second central moment of X is called the *variance* of X, and is denoted by $\mathrm{Var}[X]$. The variance of X is given by

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

The *standard deviation* of X is $\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}$. We define the *skewness* of X to be the third central moment, $\mathbb{E}[(X - \mathbb{E}[X])^3]$, and the *coefficient of skewness* to be given by

$$\mathbb{E}[(X - \mathbb{E}[X])^3]/((\mathrm{SD}[X])^3). \tag{1.1}$$

We define the *coefficient of kurtosis* of X to be

$$\mathbb{E}[(X - \mathbb{E}[X])^4]/((\mathrm{SD}[X])^4), \tag{1.2}$$

but note that various definitions are given in the literature; see the discussion in §2.2.5.

The *covariance* of random variables X and W is given by

$$\mathrm{Cov}[X, W] = \mathbb{E}[(X - \mathbb{E}[X])(W - \mathbb{E}[W])] = \mathbb{E}[XW] - \mathbb{E}[X]\mathbb{E}[W].$$

The *correlation* between random variables X and W (with $\mathrm{Var}[X] > 0$ and $\mathrm{Var}[W] > 0$) is given by

$$\mathrm{Corr}[X, W] = \frac{\mathrm{Cov}[X, W]}{\sqrt{\mathrm{Var}[X]\,\mathrm{Var}[W]}}.$$

For random variables $X_1, \ldots, X_n$ we have

$$\mathrm{Var}[X_1 + \cdots + X_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2\sum_{i<j} \mathrm{Cov}[X_i, X_j].$$

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Random variables $X_1, \dots, X_n$ are independent if, for all $x_1, \dots, x_n$ in $\mathbb{R}$,

$$\Pr(X_1 \le x_1, \dots, X_n \le x_n) = \Pr(X_1 \le x_1) \dots \Pr(X_n \le x_n).$$

For independent random variables $X_1, \dots, X_n$ and functions $h_1, \dots, h_n$, we have

$$\mathbb{E}[h_1(X_1) \dots h_n(X_n)] = \mathbb{E}[h_1(X_1)] \dots \mathbb{E}[h_n(X_n)].$$

This means that, for independent random variables $X_1, \dots, X_n$, we have

$$\mathrm{Var}[X_1 + \dots + X_n] = \mathrm{Var}[X_1] + \dots + \mathrm{Var}[X_n],$$

because, for $i \ne j$, the independence of X_i and X_j implies that $\mathrm{Cov}[X_i, X_j] = 0$. Random variables $X_1, X_2, \dots$ are independent if every finite subset of the X_i is independent. We say $X_1, X_2, \dots$ are independent and identically distributed (iid) if they are independent and all have the same distribution.

Conditioning is one of the main tools used throughout this book, and it is often the key to a neat approach to derivation of properties and features of the risk models considered in later chapters. The conditional expectation of X given W is denoted $\mathbb{E}[X \mid W]$. The very useful *conditional expectation formula* states that

$$\mathbb{E}[\mathbb{E}[X \mid W]] = \mathbb{E}[X]. \tag{1.3}$$

The *conditional variance* of X given W is defined to be

$$\begin{aligned}\mathrm{Var}[X \mid W] &= \mathbb{E}\Big[(X - \mathbb{E}[X \mid W])^2 \mid W\Big] \\ &= \mathbb{E}[X^2 \mid W] - (\mathbb{E}[X \mid W])^2.\end{aligned}$$

The *conditional variance formula* is

$$\mathrm{Var}[X] = \mathbb{E}[\,\mathrm{Var}[X \mid W]] + \mathrm{Var}\,[\mathbb{E}[X \mid W]]. \tag{1.4}$$

This may be seen by considering the terms on the right-hand side of (1.4). We have

$$\begin{aligned}\mathbb{E}\,[\mathrm{Var}[X \mid W]] &= \mathbb{E}\Big[\mathbb{E}[X^2 \mid W] - (\mathbb{E}[X \mid W])^2\Big] \\ &= \mathbb{E}[X^2] - \mathbb{E}\Big[(\mathbb{E}[X \mid W])^2\Big],\end{aligned}$$

where we have used the conditional expectation formula, and

$$\begin{aligned}\mathrm{Var}\,[\mathbb{E}[X \mid W]] &= \mathbb{E}\Big[(\mathbb{E}[X \mid W])^2\Big] - (\mathbb{E}[\mathbb{E}[X \mid W]])^2 \\ &= \mathbb{E}\Big[(\mathbb{E}[X \mid W])^2\Big] - (\mathbb{E}[X])^2,\end{aligned}$$

on using the conditional expectation formula again. Adding these terms it is easy to see that the right-hand side of (1.4) is equal to the left-hand side.

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We assume that *moment generating functions*, *probability generating functions* and their properties are familiar to the reader. The moment generating function of a random variable X is denoted

$$M_X(r) = \mathbb{E}[e^{rX}], \tag{1.5}$$

and this may not be finite for all r in $\mathbb{R}$. For every random variable X, we have $M_X(0) = 1$, and so the moment generating function is certainly finite at $r = 0$. If $M_X(r)$ is finite for $|r| < h$ for some $h > 0$, then, for any $k = 1, 2, \ldots$, the function $M_X(r)$ is k-times differentiable at $r = 0$, with

$$M_X^{(k)}(0) = \mathbb{E}[X^k], \tag{1.6}$$

with $\mathbb{E}[|X|^k]$ finite. If random variables X and W have $M_X(r) = M_W(r)$ for all $|r| < h$ for some $h > 0$, then X and W have the same distribution.

The moment generating function of a continuous random variable Y is

$$M_Y(r) = \int_{-\infty}^{\infty} e^{ry} f_Y(y)dy.$$

The moment generating function of a discrete random variable N concentrated on $\mathbb{N}$ is

$$M_N(r) = \sum_{k=0}^{\infty} e^{rk} \Pr(N = k).$$

The *probability generating function* of N is

$$G_N(z) = \mathbb{E}[z^N] = \sum_{k=0}^{\infty} z^k \Pr(N = k), \tag{1.7}$$

for those z in $\mathbb{R}$ for which the series converges absolutely. Since the series converges for $|z| \le 1$ (and possibly for a larger set of z-values), we see that the radius of convergence of the series is greater than or equal to 1. If $\mathbb{E}[N] < \infty$ then

$$\mathbb{E}[N] = G_N'(1),$$

and if $\mathbb{E}[N^2] < \infty$ then

$$\mathrm{Var}[N] = G_N''(1) + G_N'(1) - (G_N'(1))^2,$$

where $G_N^{(k)}(1) = \lim_{z\uparrow 1} G_N^{(k)}(z)$ if the radius of convergence of G_N is 1. From (1.5) and (1.7) we have

$$G_N(z) = M_N\big(\log(z)\big) \text{ and } M_N(r) = G_N(e^r),$$

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where here, and throughout the book, when we write down relationships between generating functions, we assume the phrase "for values of the argument for which both sides are finite".

Moment generating functions and probability generating functions are both examples of *transforms*. Transforms are useful for calculations involving sums of independent random variables. Let $X_1, \ldots, X_n$ be independent random variables, and let M_{X_i} be the moment generating function of X_i, $i = 1, \ldots, n$. Then the moment generating function of $T = X_1 + \cdots + X_n$ is the product of the moment generating functions of the X_i:

$$M_T(r) = M_{X_1}(r) \ldots M_{X_n}(r). \tag{1.8}$$

Similarly, let $N_1, \ldots, N_n$ be independent discrete random variables taking values in $\mathbb{N}$, and let G_{N_i} be the probability generating function of N_i, $i = 1, \ldots, n$. Then the probability generating function of $M = N_1 + \cdots + N_n$ is

$$G_M(z) = G_{N_1}(z) \ldots G_{N_n}(z). \tag{1.9}$$

Sums of independent random variables play an important role in the models in this book, so transform methods will be important for us.

The *cumulant generating function* $K_X(t)$ of a random variable X is given by

$$K_X(t) = \log(M_X(t)),$$

and this is discussed further in §2.2.5.

In the above discussion, we have given separate expectation formulae for continuous random variables and for discrete random variables. We now introduce a more general notation that covers both of these cases (and other cases as well). For a general random variable X with distribution function F_X, we write

$$\mathbb{E}[X] = \int x F_X(dx). \tag{1.10}$$

This is a Lebesgue–Stieltjes integral. We can think of the integral as shorthand notation for $\int x f_X(x) dx$ if X is continuous with density f_X, and as shorthand for $\sum_{k=0}^{\infty} k \Pr(X = k)$ if X is discrete and takes values in $\{0, 1, 2, \ldots\}$. This notation means we can give just one formula that covers both continuous and discrete random variables. However, it also covers more general random variables. Later in this book we will meet and use random variables which are neither purely continuous, nor purely discrete, but which have both a discrete part and a continuous part. To make this precise, suppose that there exist real numbers $x_1, \ldots, x_m$ and $p_1, \ldots, p_m$, where $0 \le p_k \le 1$ for $k = 1, \ldots, m$, and

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where $\sum_{k=1}^{m} p_k \le 1$, and suppose there also exists a non-negative function f, with $\int_{-\infty}^{\infty} f(t)dt \le 1$, such that the distribution function of X is

$$F_X(x) = \Pr(X \le x) = \sum_{\{k: x_k \le x\}} p_k + \int_{-\infty}^{x} f(t)dt. \tag{1.11}$$

Of course, we must have

$$\sum_{k=1}^{m} p_k + \int_{-\infty}^{\infty} f(x)dx = 1.$$

In this case, the distribution of X consists of a discrete part, specified by the x_k and the p_k (with $\Pr(X = x_k) = p_k$), and also a continuous part, specified by f. The distribution function F_X has an upward jump of size p_k at x_k, $k = 1, \ldots, m$, and is continuous and non-decreasing (and not necessarily flat) between these jumps. We say that the distribution of X has an atom at x_k (of size p_k), for $k = 1, \ldots, m$. For this X, and for a set A, we have

$$\Pr(X \in A) = \int_A F_X(dx) = \sum_{\{k: x_k \in A\}} p_k + \int_A f(x)dx. \tag{1.12}$$

As in (1.10), the expectation of X is $\mathbb{E}[X] = \int x F_X(dx)$, and, with F_X as in (1.11), the integral is

$$\int x F_X(dx) = \sum_{k=1}^{m} k p_k + \int_{-\infty}^{\infty} x f(x)dx. \tag{1.13}$$

In general, for a function h, we have

$$\mathbb{E}[h(X)] = \int h(x) F_X(dx), \tag{1.14}$$

and, when $h(x) = e^{rx}$, we find that the moment generating function of X is

$$M_X(r) = \mathbb{E}[e^{rX}] = \int e^{rx} F_X(dx). \tag{1.15}$$

With F_X as in (1.11), the equations (1.14) and (1.15) become

$$\mathbb{E}[h(X)] = \sum_{k=1}^{m} h(k) p_k + \int_{-\infty}^{\infty} h(x) f(x)dx$$

and

$$M_X(r) = \int e^{rx} F_X(dx) = \sum_{k=1}^{m} e^{rk} p_k + \int_{-\infty}^{\infty} e^{rx} f(x)dx.$$

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Note that a Lebesgue–Stieltjes integral over an interval $(a, b]$, $a \leq b$, is written

$$\int_{(a,b]} \ldots F_X(dx),$$

where ... is to be replaced by the required function to be integrated. Finally, we have, from (1.12),

$$\int_{(a,b]} F_X(dx) = \Pr\left(X \in (a, b]\right) = F_X(b) - F_X(a^-),$$

where $F_X(a^-)$ denotes $\lim_{x \to a^-} F_X(x)$, and $x \to a^-$ means that x converges to a from the left.

In this subsection, we have given a brief overview of probability. For more discussion and details, see, for example, Grimmett and Stirzaker (2001), Gut (2009) and the more advanced Gut (2005).

1.2.2 Statistics

We assume that the reader has met point estimation and properties of estimators (for example, the idea of an unbiased estimator), confidence intervals and hypothesis tests (for example, t tests, χ^2 tests, Kolmogorov–Smirnov test). We further assume a working knowledge of maximum likelihood estimators and their large sample properties. Familiarity with plots, such as histograms and quantile (or Q–Q) plots, is assumed, in addition to familiarity with the empirical distribution function. Useful references are DeGroot and Schervish (2002) and Casella and Berger (1990). The introduction to §2.4 contains an overview of some ideas and methods in statistics. At various points in the book we use more advanced statistical ideas – whenever we do this, references to appropriate texts are given.

1.2.3 Simulation

We take as prerequisite some knowledge of simulation of observations from a given distribution using a pseudo-random number generator and various techniques, such as the inverse transform (or inversion or probability integral transform) method. For more details and background, see, for example, chapter 11 in DeGroot and Schervish (2002) and chapter 6 in Morgan (2000).

1.2.4 The statistical software package *R*

The simulations, statistical analyses and numerical approximations in this book are carried out using the statistical software package ***R***. We assume familiarity

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with how ***R*** works and with basic commands in ***R***. Useful references are Venables and Ripley (2002) and Verzani (2005). The package ***R*** is available for (free) download; see `http://cran.r-project.org/`.

There is an add-on actuarial package `actuar`, and this can be installed using the `Installpackage(s)` submenu of the `Packages` menu. Choose a convenient CRAN mirror, and then select the package `actuar` for installation. It only has to be installed once, but it must be attached to the ***R*** workspace at the beginning of each ***R*** session, using the ***R*** command `library(actuar)`.

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