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978-0-521-86383-4 - Relativistic Figures of Equilibrium

Reinhard Meinel, Marcus Ansorg, Andreas Kleinwachter, Gernot Neugebauer and David Petroff
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RELATIVISTIC FIGURES OF EQUILIBRIUM

Ever since Newton introduced his theory of gravity, many famous physicists and mathematicians have worked on the problem of determining the properties of rotating bodies in equilibrium, such as planets and stars. In recent years, neutron stars and black holes have become increasingly important, and observations by astronomers and modelling by astrophysicists have reached the stage where rigorous mathematical analysis needs to be applied in order to understand their basic physics.

This book treats the classical problem of gravitational physics within Einstein's theory of general relativity. It begins by presenting basic principles and equations needed to describe rotating fluid bodies, as well as black holes in equilibrium. It then goes on to deal with a number of analytically tractable limiting cases, placing particular emphasis on the rigidly rotating disc of dust. The book concludes by considering the general case, using powerful numerical methods that are applied to various models, including the classical example of equilibrium figures of constant density.

Researchers in general relativity, mathematical physics and astrophysics will find this a valuable reference book on the topic. A related website containing codes for calculating various figures of equilibrium is available at www.cambridge.org/9780521863834.

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Preface

The theory of figures of equilibrium of rotating, self-gravitating fluids was developed in the context of questions concerning the shape of the Earth and celestial bodies. Many famous physicists and mathematicians such as Newton, Maclaurin, Jacobi, Liouville, Dirichlet, Dedekind, Riemann, Roche, Poincaré, H. Cartan, Lichtenstein and Chandrasekhar made important contributions. Within Newton's theory of gravitation, the shape of the body can be inferred from the requirement that the force arising from pressure, the gravitational force and the centrifugal force (in the corotating frame) be in equilibrium. Basic references are the books by Lichtenstein (1933) and Chandrasekhar (1969).

Our intention with the present book is to treat the *general relativistic* theory of equilibrium configurations of rotating fluids. This field of research is also motivated by astrophysics: neutron stars are so compact that Einstein's theory of gravitation must be used for calculating the shapes and other physical properties of these objects. However, as in the books mentioned above, which inspired this book to a large extent, we want to present the basic theoretical framework and will not go into astrophysical detail. We place emphasis on the rigorous treatment of simple models instead of trying to describe real objects with their many complex facets, which by necessity would lead to ephemeral and inaccurate models.

The basic equations and properties of equilibrium configurations of rotating fluids within general relativity are described in Chapter 1. We start with a discussion of the concept of an isolated body, which allows for the treatment of a single body without the need for dealing with the 'rest of the universe'. In fact, the assumption that the distant external world is *isotropic*, makes it possible to *justify* the condition of 'asymptotic flatness' in the body's far field region. Rotation 'with respect to infinity' then means nothing more than rotation with respect to the distant environment (the 'fixed stars') – very much in the spirit of Mach's principle. The main part of Chapter 1 provides a consistent mathematical formulation of the rotating fluid body problem within general relativity including its thermodynamic aspects. Conditions

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for parametric (quasi-stationary) transitions from rotating fluid bodies to black holes are also discussed.

Chapter 2 is devoted to the careful analytical treatment of limiting cases: (i) the Maclaurin spheroids, a well-known sequence of axisymmetric equilibrium configurations of homogeneous fluids in the Newtonian limit; (ii) the Schwarzschild spheres, representing non-rotating, relativistic configurations with constant mass-energy density; and (iii) the relativistic solution for a uniformly rotating disc of dust. The exact solution to the disc problem is rather involved and a detailed derivation of it will be provided here, which includes a discussion of aspects that have not been dealt with elsewhere. The solution is derived by applying the ‘inverse method’ – first used to solve the Korteweg–de Vries equation in the context of soliton theory – to Einstein’s equations. The mathematical and physical properties of the disc solution including its black hole limit (extreme Kerr metric) are discussed in some detail. At the end of Chapter 2, we show that the inverse method also allows one to *derive* the general Kerr metric as the unique solution to the Einstein vacuum equations for well-defined boundary conditions on the horizon of the black hole.

In Chapter 3, we demonstrate how one can solve general fluid body problems by means of numerical methods. We apply them to give an overview of relativistic, rotating, equilibrium configurations of constant mass-energy density. Configurations with other selected equations of state as well as ring-like bodies with a central black hole are treated summarily. A *related website* provides the reader with, amongst other things, a computer code based on a highly accurate spectral method for calculating various equilibrium figures.

Finally, we discuss some aspects of stability of equilibrium configurations and their astrophysical relevance.

We hope that our book – with its presentation of analytical *and* numerical methods – will be of value to students and researchers in general relativity, mathematical physics and astrophysics.

Acknowledgments

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Notation

Units: $G = c = 1$ (G : Newton’s gravitational constant, c : speed of light)

Complex conjugation: $\overline{a + ib} = a - ib$ (a, b real)

Greek indices ($\alpha, \beta, \dots$): run from 1 to 3

Latin indices ($a, b, \dots$): run from 1 to 4

Minkowski space: $ds^2 = \eta_{ab} dx^a dx^b = dx^2 + dy^2 + dz^2 - dt^2$
 $(x^1 = x, x^2 = y, x^3 = z, x^4 = t)$

Metric of a rotating fluid body in equilibrium:

$$ds^2 = e^{-2U} \left[e^{2k} (d\varrho^2 + d\zeta^2) + W^2 d\varphi^2 \right] - e^{2U} (dt + a d\varphi)^2$$

$$= e^{2\alpha} (d\varrho^2 + d\zeta^2) + W^2 e^{-2v} (d\varphi - \omega dt)^2 - e^{2v} dt^2$$

Killing vectors: $\xi = \partial/\partial t$ and $\eta = \partial/\partial \varphi$

Four-velocity of the fluid: $u^i = e^{-V} (\xi^i + \Omega \eta^i)$, $\Omega = \text{constant}$

Energy-momentum tensor: $T_{ik} = (\epsilon + p) u_i u_k + p g_{ik}$

Equation of state: $\epsilon = \epsilon(p)$