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Overview

The past two and a half decades have witnessed the birth and evolution of a new quantum fluid that has produced some of the most profoundly beautiful structures in physics. This fluid is formed when electrons are confined to two dimensions, cooled to near absolute zero temperature, and subjected to a strong magnetic field. In this overview chapter, results and ideas are stated without explanation; they reappear later in the book with greater elaboration.

1.1 Integral quantum Hall effect

The field began in 1980 with the discovery of the integral quantum Hall effect (IQHE), when von Klitzing observed (Fig. 1.1) plateaus in the plot of the Hall resistance as a function of the magnetic field. The Hall resistance on these plateaus is precisely quantized at

$$R_{\rm H} = \frac{h}{ne^2}, \tag{1.1}$$

where n is an integer, h is the Planck constant, and e is the electron charge. Quantizations in physics are as old as quantum mechanics itself. What is remarkable about the quantization of the Hall resistance is that it is a universal property of a complex, macroscopic system, independent of materials details, sample type or geometry; it is also robust to variation in temperature and disorder, provided they are sufficiently small. Furthermore, concurrent with the quantized plateaus is a "superflow," i.e., a dissipationless current flow in the limit of zero temperature.

It has been known since the early days of quantum mechanics that a magnetic field quantizes the kinetic energy of an electron confined in two dimensions. These quantized kinetic energy levels are called Landau levels (LLs), which are separated by a "cyclotron energy" gap. The number of filled Landau levels, which depends on the electron density (ρ) and the magnetic field (B), is called the filling factor (ν), given by

$$\nu = \frac{\rho hc}{eB}. \tag{1.2}$$

The $R_{\rm H} = h/ne^2$ plateau occurs in the magnetic field range where $\nu \approx n$.

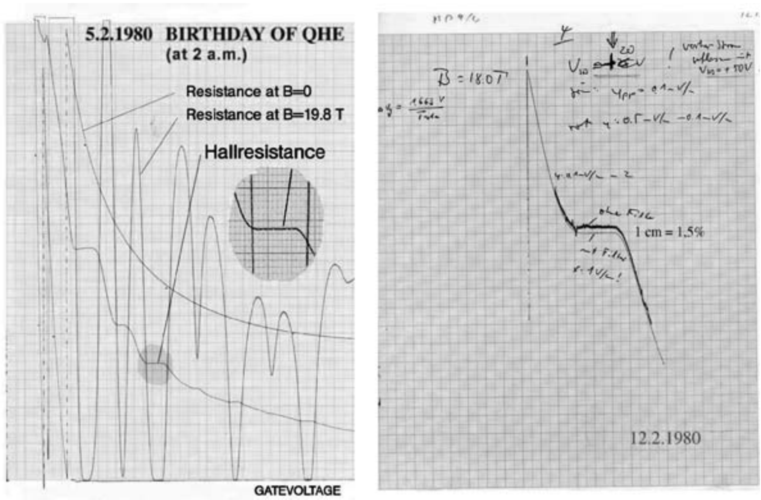

Fig. 1.1. Left: Discovery of the integral quantum Hall effect by K. von Klitzing (February 5, 1980). Right: The first high precision measurement (February 12, 1980). Courtesy: K. von Klitzing. (Reprinted with permission.)

The IQHE is a consequence of the LL quantization of the electron's kinetic energy in a magnetic field, and can be understood in terms of free electrons. When the filling factor is an integer ($\nu = n$), the ground state is especially simple: N electrons occupy the N single particle orbitals of the lowest n Landau levels, described by a wave function that is a single Slater determinant, denoted Φ_n. The state is "incompressible," i.e., its excitations cost a nonzero energy. This incompressibility at integral fillings, combined with disorder-induced Anderson localization, leads to an explanation of the IQHE [368].

1.2 Fractional quantum Hall effect

The 1982 discovery by Tsui, Stormer and Gossard of the 1/3 effect, i.e., of a quantized plateau at

$$R_{\rm H} = \frac{h}{\frac{1}{3}e^2}, \tag{1.3}$$

(Fig. 1.2) raised the field to a whole new level. This plateau is seen when the lowest Landau level is approximately 1/3 full (i.e., $\nu \approx 1/3$). Laughlin [369] stressed the correlated nature of the "1/3 state," and wrote an elegant wave function for the ground state at $\nu = 1/m$:

$$\Psi_{1/m} = \prod_{j<k} (z_j - z_k)^m \, \mathrm{e}^{-\frac{1}{4}\sum_i |z_i|^2}. \tag{1.4}$$

Here, $z_j = x_j - \mathrm{i}y_j$ denotes the two-dimensional coordinates of the jth electron as a complex number, and the magnetic length $\ell = \sqrt{\hbar c/eB}$ has been chosen as the unit of length. The

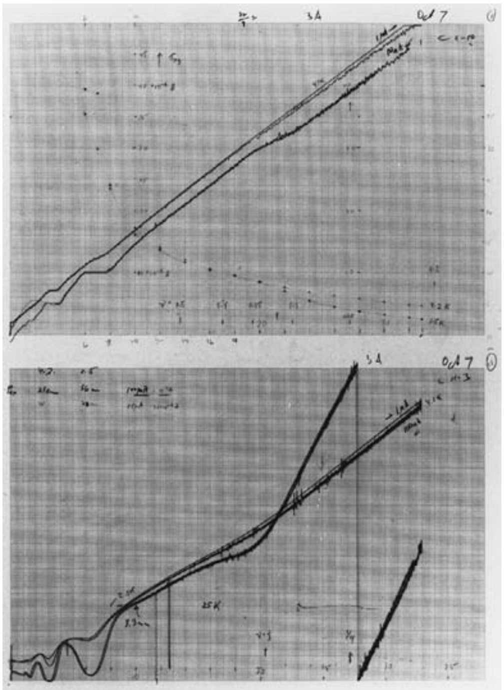

Fig. 1.2. Discovery of the fractional quantum Hall effect (FQHE) by D. C. Tsui, H. L. Stormer and A. C. Gossard. October 7, 1981. Source: H. L. Stormer, *Rev. Mod. Phys.* **71** 875–889 (1999). (Reprinted with permission.)

exponent m must be an odd integer to ensure antisymmetry. Laughlin argued that this state is incompressible, and that the excitations have a fractional charge of magnitude e/m.

The 1/3 plateau had offered only the first glimpse into an extraordinary new collective state of matter. Subsequent experimentation revealed new phenomena whose breathtaking beauty and richness would have been impossible to anticipate, by any stretch of the imagination, at the time of the first observation of $1/3$. Many more quantized plateaus,

$$R_{\mathrm{H}} = \frac{h}{fe^2}, \tag{1.5}$$

labeled by different fractions f, were observed. The number of fractions rapidly mushroomed over the years with refinements in experimental conditions, and new fractions are being discovered even now. Figure 1.3 shows a more recent magnetoresistance trace. An explanation of the physics of this new collective state of matter, which rivals superconductivity and Bose–Einstein superfluidity in both scope and elegance of the phenomena associated with it, has been an exciting challenge for, and accomplishment of, the modern many-body condensed matter theory.

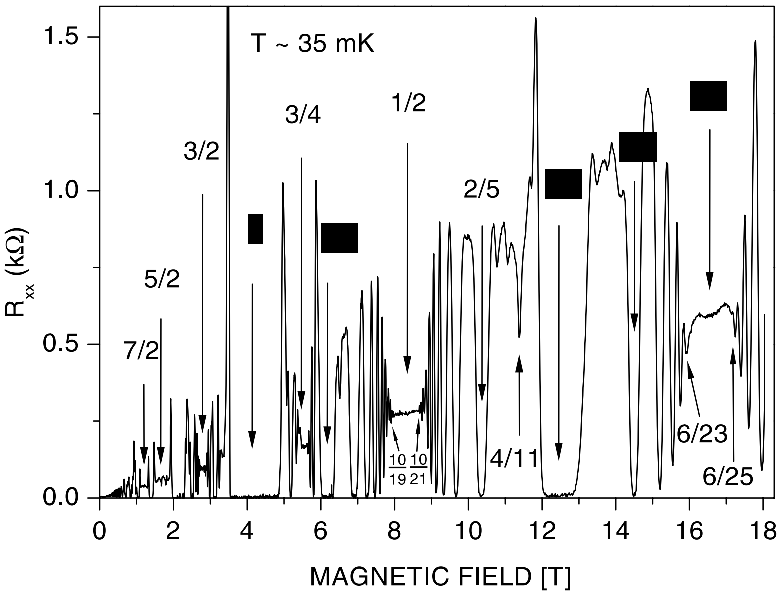

Fig. 1.3. The magnificent FQHE skyline. Diagonal resistance as a function of the magnetic field for a two-dimensional electron system with a mobility of 10 million cm^2/V s. A FQHE or an IQHE state is associated with each minimum. Many arrows only indicate the positions of filling factors (for example, 1/2, 1/4, etc.) and have no FQHE associated with them. Source: W. Pan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West, *Phys. Rev. Lett.* **88**, 176802 (2002). (Reprinted with permission.)

1.3 Strongly correlated state

At high magnetic fields all electrons occupy the lowest Landau level. Their kinetic energy is then constant, hence irrelevant. Interacting electrons in a high magnetic field are mathematically described by the Hamiltonian

$$H = \sum_{j<k} \frac{1}{r_{jk}} \qquad \text{(lowest Landau level)}, \tag{1.6}$$

which is to be solved in the lowest Landau level (LLL) subspace. (The quantity $r_{jk} = |z_j - z_k|$ is the distance between the electrons j and k.) The "pure" FQHE problem has no parameters. The Hamiltonian looks simple until one makes the following (related) observations:

(i) **The no-small-parameter problem** The theory contains no parameters. (The Coulomb interaction merely sets the overall energy scale.)

(ii) **The degeneracy problem** The number of degenerate ground states in the absence of interaction (with $H = 0$) is astronomically large.

(iii) **The no-normal-state problem** In some instances, a nontrivial collective phenomenon can be understood as an instability of a "normal state," which is the state that would be obtained if the

> interaction could be switched off. For the FQHE problem, switching off the interaction does not produce a meaningful state. The FQHE has no normal state. The physics of the FQHE state is "strongly" non-perturbative, in the sense that no weak coupling limit exists in which the solution is close to a known solution. We do not have the luxury of calculating small deviations from a normal state, but must determine "full" answers.

We wish to identify the principle responsible for the dramatic physics revealed by experiment, but do not even know where to begin. Yet this has turned out to be a theorist's dream problem, in which, thanks to an abundance of experimental clues and an incredible amount of luck, a precise and secure understanding has been achieved. A new language has been developed to describe the concepts governing the behavior of this state. Many facets of the FQHE physics, as well as numerous related phenomena, follow directly from a single unifying principle: the formation of "composite fermions."

1.4 Composite fermions

With the observation of many fractions, an analogy with the IQHE could be identified, which has led to an explanation of the FQHE (and more). The eigenfunctions and eigenenergies for the ground and (low-energy) excited states of strongly interacting electrons at an arbitrary LLL filling ν are expressed in terms of the known solutions of the noninteracting electron problem at the LL filling ν^* as follows:

$$\Psi_\nu = \mathcal{P}_{\rm LLL} \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\nu^*}, \tag{1.7}$$

and

$$E_\nu = \frac{\langle \Psi_\nu | \sum_{j<k} \frac{1}{r_{jk}} | \Psi_\nu \rangle}{\langle \Psi_\nu | \Psi_\nu \rangle} + V_{\rm el-bg} + V_{\rm bg-bg}, \tag{1.8}$$

where

$$\nu = \frac{\nu^*}{2p\nu^* \pm 1}, \tag{1.9}$$

$$B^* = B - 2p\rho\phi_0. \tag{1.10}$$

Here, Φ_{ν^*} are the eigenfunctions of noninteracting electrons at ν^*, $\mathcal{P}_{\rm LLL}$ projects the wave function to its right into the lowest Landau level, p is an integer, B^* is an "effective" magnetic field, ρ is the two-dimensional density, $\phi_0 = hc/e$ is called the flux quantum, and $V_{\rm el-bg}$ and $V_{\rm bg-bg}$ are electron–background and background–background interaction energies (assuming a uniform positively charged neutralizing background). These equations define a one-to-one correspondence between the ground and excited states at filling factor ν (or magnetic field B) and ν^* (magnetic field B^*). They have the following physical interpretation (which is more generally valid than Eqs. (1.7) and (1.8)):

- The Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$ attaches $2p$ quantized vortices to each electron in Φ_{ν^*}. The bound state of an electron and $2p$ vortices is interpreted as a particle called the "composite fermion."

It is sometimes pictured and modeled as the bound state of an electron and an even number of magnetic flux quanta, although that picture should not be taken literally. Electrons capture vortices to turn into composite fermions because that is how they minimize their interaction energy.

- As composite fermions move about, the vortices bound to them produce Berry phases, which cancel part of the Aharonov–Bohm phases originating from the external magnetic field. Composite fermions thus sense a magnetic field B^* that is much smaller than the applied magnetic field, and can even be zero. This property of composite fermions distinguishes them from electrons, and lies at the root of most of the dramatic phenomenology. Composite fermions form Landau-like levels (called Λ levels) in the reduced magnetic field, and occupy ν^* of them.
- The right hand side of the expression for Ψ_ν (Eq. 1.7) is interpreted as the wave function of composite fermions at filling factor ν^*.

Detailed quantitative calculations have shown Eqs. (1.7) and (1.8) to be accurate, and experiments have confirmed their interpretation in terms of composite fermions. Composite fermions have been directly observed in many experiments and their numerous consequences have been verified in repeated tests over the last decade and a half. Composite fermions embody the nonperturbative reorganization that takes place when a collection of two-dimensional electrons is subjected to a strong magnetic field.

Equations (1.7) and (1.8) represent the "quantum mechanical wave function" formulation of composite fermions. The physics of composite fermions has also been implemented through other calculational frameworks, most notably a topological field theory known as the composite fermion Chern–Simons (CFCS) theory. The Laughlin wave function is recovered as a special case of Eq. (1.7) with $\nu^* = 1$, $2p = m - 1$, and with Φ_1 taken as the ground state at filling factor one.

Composite fermions are bizarre particles in many respects. They represent a new class of particles realized in nature. Previously known fermions were either elementary fermions or their bound states. A composite fermion, on the other hand, is the bound state of an electron and an even number of quantized vortices. The vortex is a collective, topological, quantum object. It is not a degree of freedom in the Hamiltonian but an emergent state of *all* electrons; it has quantum mechanical phases associated with it; and it is a topological entity because the quantum mechanical phase associated with a closed loop around a vortex is exactly 2π, independent of the shape and the size of the loop. (The topological character of vortices is implicit in the fact that we *count* them.) As a result, composite fermions are collective, topological, quantum particles. We note: (a) Even a single composite fermion is a collective bound state of all electrons. It is surprising that composite fermions behave as almost free, ordinary particles to a great extent. (b) The quantum mechanical phases associated with the vortex give the composite fermion an inherently quantum mechanical character. While quantum mechanics describes all particles, it is responsible for the very creation of composite fermions. A purely classical world would have no composite fermions. (c) All fluids of composite fermions are topological quantum fluids. The topological quantization of the vorticity of composite fermions is responsible for Hall quantization, the effective magnetic field, and numerous other phenomena. The emergence of such a complex particle is a testament to the genuinely collective character of this quantum fluid.

1.5 Origin of the FQHE

The formation of composite fermions leads to an immediate understanding of the enigmatic FQHE of electrons as a manifestation of the IQHE of composite fermions. The latter occurs when the CF filling factor is an integer, i.e., $\nu^* = n$. The integral fillings of composite fermions correspond to electron filling factors given by

$$\nu = \frac{n}{2pn \pm 1}, \tag{1.11}$$

which nicely match the prominently observed fractions. This explanation clarifies:

- the origin of gaps at certain fractional fillings (composite fermions fill n Landau-like levels);
- appearance of sequences of fractions (fractions are derived from the sequence of integers);
- their order of stability;
- the abundance of odd-denominator fractions;
- exactness of the Hall quantization (the right hand side of Eq. (1.11) is made up of whole numbers, hence invariant under weak perturbations; the integral value of p has a topological origin).

At relatively low magnetic fields, when the spin degree of freedom is not frozen, several FQHE states with different spin polarizations become possible at each of the fractions in Eq. (1.11). These are explained in terms of composite fermions carrying spin.

1.6 The composite fermion quantum fluid

In addition to explaining the FQHE at the fractions in Eq. (1.11), composite fermions open the door into a new world where many other phenomena can be studied experimentally and explained or predicted theoretically. Some notable states of composite fermions are (also including their IQHE for completeness):[1]

- IQHE of composite fermions [$\nu = n/(2pn \pm 1)$]
- composite fermion Fermi sea ($\nu = 1/2$)
- Bardeen–Cooper–Schrieffer-like p-wave paired state ($\nu = 5/2$)
- FQHE of composite fermions (e.g., $\nu = 4/11$)
- quantum crystal of composite fermions*
- composite fermion stripes and bubble crystals*

Many phenomena of composite fermions (other than the IQHE and the FQHE) have been measured:

- Shubnikov–de Haas oscillations
- commensurability oscillations
- surface acoustic wave absorption

[1] The items marked by an asterisk have not been observed experimentally yet.

- thermopower
- magnetic focusing
- bilayer drag
- compressibility

These have enabled a determination of the quantum numbers and other parameters of composite fermions:

- charge
- spin
- exchange statistics
- effective mass
- magnetic moment
- Fermi wave vector
- cyclotron orbits

Composite fermions have been directly observed in geometric experiments, where they can be manipulated like billiard balls, injected at one point and collected at another. Such experiments have measured trajectories of the current-carrying objects and found them to be consistent with the effective magnetic field, confirming that the current carriers are not electrons but composite fermions.

Additionally, many excitations of composite fermions have been observed:

- charged excitations
- excitons
- rotons
- bi-rotons
- skyrmions
- spin-flip excitations
- cyclotron resonance
- flavor-altering excitations

Excited composite fermions are called CF-quasiparticles. The topological character of composite fermions results in the following unusual properties for the CF-quasiparticles:

- Fractional "local" charge: The charge excess associated with a CF-quasiparticle is a precise fraction of an electron charge.
- Fractional "braiding" statistics: CF-quasiparticles are believed to acquire, theoretically, nontrivial phases under braiding around one another. They are currently the most promising candidate for a physical realization of "anyons," i.e., particles obeying fractional braiding statistics.

Experimental evidence exists for fractional charge. The braiding property has not yet been verified experimentally.

1.7 An "ideal" theory

The composite fermion theory possesses many qualities desirable in a theory.

Unification It explains very many experimental facts starting from a single principle. It obtains all fractions in an equivalent manner, unifies the fractional and the integral quantum Hall effects, and describes states that do not exhibit FQHE (for example, the compressible states at even denominator fractions). It gives a unified theory of quasiparticles, quasiholes, and various other excitations of FQHE states.

Simplicity It provides simple intuitive explanations for the basic phenomenology of the fractional quantum Hall effect, e.g., for the appearance of certain sequences of odd-denominator fractions, and the lack of FQHE at even-denominator fractions (with one exception).

Uniqueness An aim of theoretical physics is to reduce the number of adjustable parameters to the maximum extent possible. The composite fermion principle is "super-constraining" in that it provides parameter-free answers for many quantities of interest. In particular, it uniquely identifies wave functions for the incompressible ground states and their low-energy excitations.

Accuracy Rigorous, unbiased and detailed tests against exact "computer experiments" on finite systems have shown that the CF theory gives a faithful counting of the low-energy eigenstates, and that the wave functions of Eq. (1.7) are practically exact: They typically have close to 100% overlap with the exact eigenstates, and their energies are accurate to better than 0.1%. This level of accuracy is especially noteworthy in view of the absence of adjustable parameters in the theory.

Microscopic theory Condensed matter theories aspire to explain macroscopic phenomena by solving the microscopic Hamiltonian describing a collection of interacting electrons and ions. In practice, however, the path from the microscopic Hamiltonian to the macroscopic phenomenology often passes through one or more layers of approximation that can be justified only with the help of detailed testing against experiment. The enormously successful BCS theory, for example, is built upon the crucial assumption that the normal state can be modeled in terms of weakly interacting Landau quasiparticles, which are the variables in terms of which the BCS wave function is expressed.[2] In contrast, the wave functions of the CF theory are the actual wave functions that occur in the laboratory. They are written in terms of the real electron coordinates, and represent the solutions of the Hamiltonian of Eq. (1.6) in a quantum chemistry sense.

Falsifiability Many sharp and nontrivial consequences and predictions stemming from the CF theory have been verified during the past 1.5 decades. Predictions play a more

[2] A lack of such simplification lies at the root of the difficulties encountered in the explanation of high-temperature superconductivity.

important role in the establishment of a theory than explanations of known facts. Many nontrivial predictions of the CF theory, of both qualitative and quantitative natures, have been confirmed.

New particles Emergent physics often expresses itself through new particles. In spite of their rather complex and unconventional nature, composite fermions behave as legitimate particles with mass, charge, spin, statistics, and other properties that we associate with particles. Composite fermions have numerous other particle-like manifestations; they exhibit many phenomena that electrons do.[3]

1.8 Miscellaneous remarks

The overview concludes with some other interesting facts about the CF liquid.

Macroscopic quantum phenomena without BEC We instinctively associate one or another kind of Bose–Einstein condensation (BEC) with the phrase "macroscopic quantum phenomenon," be it a ^{4}He superfluid or a BCS superconductor (the latter being, crudely, a BEC of Cooper pairs). These systems are characterized by the formation of a macroscopic quantum state with phase rigidity. The CF fluid constitutes a distinct paradigm for macroscopic quantum behavior. It *is* a quantum fluid, because quantum mechanical phases play a central role in determining its macroscopic behavior. These phases enter, however, in the form of quantized vortices, which are captured by electrons to produce new physics.

Order without order parameter The order in the CF fluid is characterized by the formation of composite fermions. There is no off-diagonal long-range order in the CF fluid, nor an order parameter in the sense defined by Landau. The CF fluid is a topological quantum fluid. The topological nature of the state is evidenced most directly through the effective magnetic field, which is responsible for the FQHE, the CF Fermi sea, and numerous other phenomena.

"Meissner effect" without induced currents In a "type-I" superconductor, an applied magnetic field is fully screened, known as the Meissner effect. Macroscopic currents are induced that produce a magnetic field of their own that exactly cancels the external field. The partial "cancellation" of the external magnetic field by the CF fluid is not caused by an induced current, but by induced Berry phases, and has a purely quantum mechanical origin. The cancellation is totally "internal" to composite fermions. An external magnetometer will assess the full applied magnetic field, but when composite fermions themselves are used to measure the magnetic field, the reduced effective magnetic field is detected.

[3] "They're as real as Cooper pairs." H. L. Stormer [623]