OPTIMIZATION METHODS IN FINANCE

Optimization models are playing an increasingly important role in financial decisions. This is the first textbook devoted to explaining how recent advances in optimization models, methods and software can be applied to solve problems in computational finance ranging from asset allocation to risk management, from option pricing to model calibration more efficiently and more accurately. Chapters discussing the theory and efficient solution methods for all major classes of optimization problems alternate with chapters illustrating their use in modeling problems of mathematical finance.

The reader is guided through the solution of asset/liability cash flow matching using linear programming techniques, which are also used to explain asset pricing and arbitrage. Volatility estimation is discussed using nonlinear optimization models. Quadratic programming formulations are provided for portfolio optimization problems based on a mean-variance model, for returns-based style analysis and for risk-neutral density estimation. Conic optimization techniques are introduced for modeling volatility constraints in asset management and for approximating covariance matrices. For constructing an index fund, the authors use an integer programming model. Option pricing is presented in the context of dynamic programming and so is the problem of structuring asset backed securities. Stochastic programming is applied to asset/liability management, and in this context the notion of Conditional Value at Risk is described. The final chapters are devoted to robust optimization models in finance.

The book is based on Master’s courses in financial engineering and comes with worked examples, exercises and case studies. It will be welcomed by applied mathematicians, operational researchers and others who work in mathematical and computational finance and who are seeking a text for self-learning or for use with courses.

Gerard Cornuejols is an IBM University Professor of Operations Research at the Tepper School of Business, Carnegie Mellon University

Reha Tüünçu is a Vice President in the Quantitative Resources Group at Goldman Sachs Asset Management, New York
Mathematics, Finance and Risk

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OPTIMIZATION METHODS IN FINANCE

GERARD CORNUEJOLS
Carnegie Mellon University

REHA TÜTÜNCÜ
Goldman Sachs Asset Management
To Julie
and
to Paz
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foreword</td>
<td>xi</td>
</tr>
<tr>
<td>1  Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Optimization problems</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Optimization with data uncertainty</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Financial mathematics</td>
<td>8</td>
</tr>
<tr>
<td>2  Linear programming: theory and algorithms</td>
<td>15</td>
</tr>
<tr>
<td>2.1 The linear programming problem</td>
<td>15</td>
</tr>
<tr>
<td>2.2 Duality</td>
<td>17</td>
</tr>
<tr>
<td>2.3 Optimality conditions</td>
<td>21</td>
</tr>
<tr>
<td>2.4 The simplex method</td>
<td>23</td>
</tr>
<tr>
<td>3  LP models: asset/liability cash-flow matching</td>
<td>41</td>
</tr>
<tr>
<td>3.1 Short-term financing</td>
<td>41</td>
</tr>
<tr>
<td>3.2 Dedication</td>
<td>50</td>
</tr>
<tr>
<td>3.3 Sensitivity analysis for linear programming</td>
<td>53</td>
</tr>
<tr>
<td>3.4 Case study: constructing a dedicated portfolio</td>
<td>60</td>
</tr>
<tr>
<td>4  LP models: asset pricing and arbitrage</td>
<td>62</td>
</tr>
<tr>
<td>4.1 Derivative securities and the fundamental theorem of asset pricing</td>
<td>62</td>
</tr>
<tr>
<td>4.2 Arbitrage detection using linear programming</td>
<td>69</td>
</tr>
<tr>
<td>4.3 Additional exercises</td>
<td>71</td>
</tr>
<tr>
<td>4.4 Case study: tax clientele effects in bond portfolio management</td>
<td>76</td>
</tr>
<tr>
<td>5  Nonlinear programming: theory and algorithms</td>
<td>80</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>80</td>
</tr>
<tr>
<td>5.2 Software</td>
<td>82</td>
</tr>
<tr>
<td>5.3 Univariate optimization</td>
<td>82</td>
</tr>
<tr>
<td>5.4 Unconstrained optimization</td>
<td>92</td>
</tr>
<tr>
<td>5.5 Constrained optimization</td>
<td>100</td>
</tr>
<tr>
<td>5.6 Nonsmooth optimization: subgradient methods</td>
<td>110</td>
</tr>
</tbody>
</table>
# Contents

6 **NLP models: volatility estimation**
   6.1 Volatility estimation with GARCH models 112
   6.2 Estimating a volatility surface 116

7 **Quadratic programming: theory and algorithms**
   7.1 The quadratic programming problem 121
   7.2 Optimality conditions 122
   7.3 Interior-point methods 124
   7.4 QP software 135
   7.5 Additional exercises 136

8 **QP models: portfolio optimization**
   8.1 Mean-variance optimization 138
   8.2 Maximizing the Sharpe ratio 155
   8.3 Returns-based style analysis 158
   8.4 Recovering risk-neutral probabilities from options prices 161
   8.5 Additional exercises 165
   8.6 Case study: constructing an efficient portfolio 167

9 **Conic optimization tools**
   9.1 Introduction 168
   9.2 Second-order cone programming 169
   9.3 Semidefinite programming 173
   9.4 Algorithms and software 177

10 **Conic optimization models in finance**
   10.1 Tracking error and volatility constraints 178
   10.2 Approximating covariance matrices 181
   10.3 Recovering risk-neutral probabilities from options prices 185
   10.4 Arbitrage bounds for forward start options 187

11 **Integer programming: theory and algorithms**
   11.1 Introduction 192
   11.2 Modeling logical conditions 193
   11.3 Solving mixed integer linear programs 196

12 **Integer programming models: constructing an index fund**
   12.1 Combinatorial auctions 212
   12.2 The lockbox problem 213
   12.3 Constructing an index fund 216
   12.4 Portfolio optimization with minimum transaction levels 222
   12.5 Additional exercises 223
   12.6 Case study: constructing an index fund 224
## 13 Dynamic programming methods

13.1 Introduction 225
13.2 Abstraction of the dynamic programming approach 233
13.3 The knapsack problem 236
13.4 Stochastic dynamic programming 238

## 14 DP models: option pricing

14.1 A model for American options 240
14.2 Binomial lattice 242

## 15 DP models: structuring asset-backed securities

15.1 Data 250
15.2 Enumerating possible tranches 252
15.3 A dynamic programming approach 253
15.4 Case study: structuring CMOs 254

## 16 Stochastic programming: theory and algorithms

16.1 Introduction 255
16.2 Two-stage problems with recourse 256
16.3 Multi-stage problems 258
16.4 Decomposition 260
16.5 Scenario generation 263

## 17 Stochastic programming models: Value-at-Risk and Conditional Value-at-Risk

17.1 Risk measures 271
17.2 Minimizing CVaR 274
17.3 Example: bond portfolio optimization 276

## 18 Stochastic programming models: asset/liability management

18.1 Asset/liability management 279
18.2 Synthetic options 285
18.3 Case study: option pricing with transaction costs 288

## 19 Robust optimization: theory and tools

19.1 Introduction to robust optimization 292
19.2 Uncertainty sets 293
19.3 Different flavors of robustness 295
19.4 Tools and strategies for robust optimization 302

## 20 Robust optimization models in finance

20.1 Robust multi-period portfolio selection 306
20.2 Robust profit opportunities in risky portfolios 311
20.3 Robust portfolio selection 313
20.4 Relative robustness in portfolio selection 315
20.5 Moment bounds for option prices 317
20.6 Additional exercises 318
<table>
<thead>
<tr>
<th>Appendix A</th>
<th>Convexity</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>Appendix B</td>
<td>Cones</td>
<td>322</td>
</tr>
<tr>
<td>Appendix C</td>
<td>A probability primer</td>
<td>323</td>
</tr>
<tr>
<td>Appendix D</td>
<td>The revised simplex method</td>
<td>327</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>338</td>
</tr>
<tr>
<td>Index</td>
<td></td>
<td>342</td>
</tr>
</tbody>
</table>
The use of sophisticated mathematical tools in modern finance is now commonplace. Researchers and practitioners routinely run simulations or solve differential equations to price securities, estimate risks, or determine hedging strategies. Some of the most important tools employed in these computations are optimization algorithms. Many computational finance problems ranging from asset allocation to risk management, from option pricing to model calibration, can be solved by optimization techniques. This book is devoted to explaining how to solve such problems efficiently and accurately using recent advances in optimization models, methods, and software.

Optimization is a mature branch of applied mathematics. Typical optimization problems have the objective of allocating limited resources to alternative activities in order to maximize the total benefit obtained from these activities. Through decades of intensive and innovative research, fast and reliable algorithms and software have become available for many classes of optimization problems. Consequently, optimization is now being used as an effective management and decision-support tool in many industries, including the financial industry.

This book discusses several classes of optimization problems encountered in financial models, including linear, quadratic, integer, dynamic, stochastic, conic, and robust programming. For each problem class, after introducing the relevant theory (optimality conditions, duality, etc.) and efficient solution methods, we discuss several problems of mathematical finance that can be modeled within this problem class. The reader is guided through the solution of asset/liability cash-flow matching using linear programming techniques, which are also used to explain asset pricing and arbitrage. Volatility estimation is discussed using nonlinear optimization models. Quadratic programming formulations are provided for portfolio optimization problems based on a mean-variance model for returns-based style analysis and for risk-neutral density estimation. Conic optimization techniques are introduced for modeling volatility constraints in asset management and for approximating
covariance matrices. For constructing an index fund, we use an integer programming model. Option pricing is presented in the context of dynamic programming and so is the problem of structuring asset-backed securities. Stochastic programming is applied to asset/liability management, and in this context the notion of Conditional Value at Risk is described. Robust optimization models for portfolio selection and option pricing are also discussed.

This book is intended as a textbook for Master’s programs in financial engineering, finance, or computational finance. In addition, the structure of chapters, alternating between optimization methods and financial models that employ these methods, allows the use of this book as a primary or secondary text in upper level undergraduate or introductory graduate courses in operations research, management science, and applied mathematics.

Optimization algorithms are sophisticated tools and the relationship between their inputs and outputs is sometimes opaque. To maximize the value one gets from these tools and to understand how they work, users often need a significant amount of guidance and practical experience with them. This book aims to provide this guidance and serve as a reference tool for the finance practitioners who use or want to use optimization techniques.

This book has its origins in courses taught at Carnegie Mellon University in the Masters program in Computational Finance and in the MBA program at the Tepper School of Business (Gérard Cornuéjols), and at the Tokyo Institute of Technology, Japan, and the University of Coimbra, Portugal (Reha Tütüncü). We thank the attendants of these courses for their feedback and for many stimulating discussions. We would also like to thank the colleagues who provided the initial impetus for this project or collaborated with us on various research projects that are reflected in the book, especially Rick Green, Raphael Hauser, John Hooker, Mark Koenig, Masakazu Kojima, Vijay Krishnamurthy, Yanjun Li, Ana Margarida Monteiro, Mustafa Pınar, Sanjay Srivastava, Michael Trick, and Luís Vicente. Various drafts of this book were experimented with in class by Javier Peña, François Margot, Miguel Lejeune, Miroslav Karamanov, and Kathie Cameron, and we thank them for their comments. Initial drafts of this book were completed when the second author was on the faculty of the Department of Mathematical Sciences at Carnegie Mellon University; he gratefully acknowledges their financial support.