1 Basic tools

1.1 The Fourier transform

Synopsis

The **Fourier transform** of $f(x)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs}\, dx$$

The inverse Fourier transform is given by

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{+i2\pi xs}\, ds$$

Evenness and oddness

A function $E(x)$ is *even* if $E(x) = E(-x)$. A function $O(x)$ is *odd* if $O(x) = -O(-x)$.

The Fourier transform has the following properties for even and odd functions:

- *Even functions*. The Fourier transform of an even function is even. A *real even* function transforms to a *real even* function. An *imaginary even* function transforms to an *imaginary even* function.
- *Odd functions*. The Fourier transform of an odd function is odd. A *real odd* function transforms to an *imaginary odd* function. An *imaginary odd* function transforms to a *real odd* function (i.e., the "realness" flips when the Fourier transform of an odd function is taken).

real even (RE) → real even (RE)
imaginary even (IE) → imaginary even (IE)
real odd (RO) → imaginary odd (IO)
imaginary odd (IO) → real odd (RO)

Any function can be expressed in terms of its even and odd parts:

$$f(x) = E(x) + O(x)$$

where

$$E(x) = \tfrac{1}{2}[f(x) + f(-x)]$$

$$O(x) = \tfrac{1}{2}[f(x) - f(-x)]$$

Then, for an arbitrary complex function we can summarize these relations as (Bracewell, 1965)

$$f(x) = \mathrm{re}(x) + i\,\mathrm{ie}(x) + \mathrm{ro}(x) + i\,\mathrm{io}(x)$$

$$F(x) = \mathrm{RE}(s) + i\,\mathrm{IE}(s) + \mathrm{RO}(s) + i\,\mathrm{IO}(s)$$

As a consequence, a real function $f(x)$ has a Fourier transform that is *hermitian,* $F(s) = F^*(-s)$, where $*$ refers to the complex conjugate.

For a more general complex function, $f(x)$, we can tabulate some additional properties (Bracewell, 1965):

$$\begin{aligned}
f(x) &\Leftrightarrow F(s)\\
f^*(x) &\Leftrightarrow F^*(-s)\\
f^*(-x) &\Leftrightarrow F^*(s)\\
f(-x) &\Leftrightarrow F(-s)\\
2\,\mathrm{Re} f(x) &\Leftrightarrow F(s) + F^*(-s)\\
2\,\mathrm{Im} f(x) &\Leftrightarrow F(s) - F^*(-s)\\
f(x) + f^*(-x) &\Leftrightarrow 2\,\mathrm{Re} F(s)\\
f(x) - f^*(-x) &\Leftrightarrow 2\,\mathrm{Im} F(s)
\end{aligned}$$

The **convolution** of two functions $f(x)$ and $g(x)$ is

$$f(x) * g(x) = \int_{-\infty}^{+\infty} f(z)\, g(x-z)\, dz = \int_{-\infty}^{+\infty} f(x-z)\, g(z)\, dz$$

Convolution theorem

If $f(x)$ has the Fourier transform $F(s)$, and $g(x)$ has the Fourier transform $G(s)$, then the Fourier transform of the convolution $f(x) * g(x)$ is the product $F(s)\,G(s)$.

The **cross-correlation** of two functions $f(x)$ and $g(x)$ is

$$f^*(x) \star g(x) = \int_{-\infty}^{+\infty} f^*(z-x)\, g(z)\, dz = \int_{-\infty}^{+\infty} f^*(z)\, g(z+x)\, dz$$

where f^* refers to the complex conjugate of f. When the two functions are the same, $f^*(x) \bigstar f(x)$ is called the **autocorrelation** of $f(x)$.

Energy spectrum

The modulus squared of the Fourier transform $|F(s)|^2 = F(s)\,F^*(s)$ is sometimes called the **energy spectrum** or simply the **spectrum**.

If $f(x)$ has the Fourier transform $F(s)$, then the autocorrelation of $f(x)$ has the Fourier transform $|F(s)|^2$.

Phase spectrum

The Fourier transform $F(s)$ is most generally a complex function, which can be written as

$$F(s) = |F|e^{i\varphi} = \mathrm{Re}\,F(s) + i\,\mathrm{Im}\,F(s)$$

where $|F|$ is the modulus and φ is the **phase**, given by

$$\varphi = \tan^{-1}[\mathrm{Im}\,F(s)/\mathrm{Re}\,F(s)]$$

The function $\varphi(s)$ is sometimes also called the **phase spectrum**.

Obviously, both the modulus and phase must be known to completely specify the Fourier transform $F(s)$ or its transform pair in the other domain, $f(x)$. Consequently, an infinite number of functions $f(x) \Leftrightarrow F(s)$ are consistent with a given spectrum $|F(s)|^2$.

The **zero-phase** equivalent function (or zero-phase equivalent wavelet) corresponding to a given spectrum is

$$F(s) = |F(s)|$$
$$f(x) = \int_{-\infty}^{\infty} |F(s)|\, e^{+i2\pi xs}\, ds$$

which implies that $F(s)$ is real and $f(x)$ is hermitian. In the case of zero-phase *real* wavelets, then, both $F(s)$ and $f(x)$ are real even functions.

The **minimum-phase** equivalent function or wavelet corresponding to a spectrum is the unique one that is both *causal* and *invertible*. A simple way to compute the minimum-phase equivalent of a spectrum $|F(s)|^2$ is to perform the following steps (Claerbout, 1992):

(1) Take the logarithm, $B(s) = \ln |F(s)|$.
(2) Take the Fourier transform, $B(s) \Rightarrow b(x)$.
(3) Multiply $b(x)$ by zero for $x < 0$ and by 2 for $x > 0$. If done numerically, leave the values of b at zero and the Nyquist frequency unchanged.
(4) Transform back, giving $B(s) + i\varphi(s)$, where φ is the desired phase spectrum.
(5) Take the complex exponential to yield the minimum-phase function: $F_{\mathrm{mp}}(s) = \exp[B(s) + i\varphi(s)] = |F(s)|e^{i\varphi(s)}$.
(6) The causal minimum-phase wavelet is the Fourier transform of $F_{\mathrm{mp}}(s) \Rightarrow f_{\mathrm{mp}}(x)$.

Another way of saying this is that the phase spectrum of the minimum-phase equivalent function is the Hilbert transform (see Section 1.2 on the Hilbert transform) of the log of the energy spectrum.

Sampling theorem

A function $f(x)$ is said to be *band limited* if its Fourier transform is nonzero only within a finite range of frequencies, $|s| < s_c$, where s_c is sometimes called the *cut-off frequency*. The function $f(x)$ is fully specified if sampled at equal spacing not exceeding $\Delta x = 1/(2s_c)$. Equivalently, a time series sampled at interval Δt adequately describes the frequency components out to the *Nyquist frequency* $f_N = 1/(2\Delta t)$.

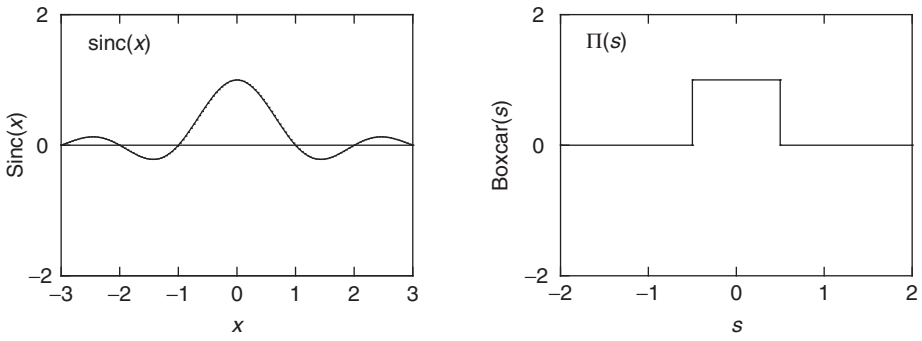

Figure 1.1.1 Plots of the function sinc(x) and its Fourier transform $\Pi(s)$.

The numerical process to recover the intermediate points between samples is to convolve with the *sinc function*:

$$2s_c\text{sinc}(2s_c x) = 2s_c\sin(\pi 2s_c x)/\pi 2s_c x$$

where

$$\text{sinc}(x) \equiv \frac{\sin(\pi x)}{\pi x}$$

which has the properties:

$$\left.\begin{array}{l}\text{sinc}(0) = 1\\ \text{sinc}(n) = 0\end{array}\right\} \; n = \text{nonzero integer}$$

The Fourier transform of sinc(x) is the boxcar function $\Pi(s)$:

$$\Pi(s) = \begin{cases} 0 & |s| > \frac{1}{2} \\ ½ & |s| = \frac{1}{2} \\ 1 & |s| < \frac{1}{2} \end{cases}$$

Plots of the function sinc(x) and its Fourier transform $\Pi(s)$ are shown in Figure 1.1.1.

One can see from the convolution and similarity theorems below that convolving with $2s_c\,\text{sinc}(2s_c x)$ is equivalent to multiplying by $\Pi(s/2s_c)$ in the frequency domain (i.e., zeroing out all frequencies $|s| > s_c$ and passing all frequencies $|s| < s_c$.

Numerical details

Consider a band-limited function $g(t)$ sampled at N points at equal intervals: $g(0)$, $g(\Delta t)$, $g(2\Delta t), \ldots, g((N-1)\Delta t)$. A typical fast Fourier transform (FFT) routine will yield N equally spaced values of the Fourier transform, $G(f)$, often arranged as

1	2	3	$\cdots$	$\left(\frac{N}{2}+1\right)$	$\left(\frac{N}{2}+2\right)$	$\cdots$	$(N-1)$	N
$G(0)$	$G(\Delta f)$	$G(2\Delta f)$	$\cdots$	$G(\pm f_N)$	$G(-f_N+\Delta f)$	$\cdots$	$G(-2\Delta f)$	$G(-\Delta f)$

time domain sample rate Δt
Nyquist frequency $f_N = 1/(2\Delta t)$
frequency domain sample rate $\Delta f = 1/(N\Delta t)$

Note that, because of "wraparound," the sample at $(N/2 + 1)$ represents both $\pm f_N$.

Spectral estimation and windowing

It is often desirable in rock physics and seismic analysis to estimate the spectrum of a wavelet or seismic trace. The most common, easiest, and, in some ways, the worst way is simply to chop out a piece of the data, take the Fourier transform, and find its magnitude. The problem is related to sample length. If the true data function is $f(t)$, a small sample of the data can be thought of as

$$f_{\text{sample}}(t) = \begin{cases} f(t), & a \le t \le b \\ 0, & \text{elsewhere} \end{cases}$$

or

$$f_{\text{sample}}(t) = f(t)\,\Pi\left(\frac{t - \frac{1}{2}(a+b)}{b-a}\right)$$

where $\Pi(t)$ is the boxcar function discussed above. Taking the Fourier transform of the data sample gives

$$F_{\text{sample}}(s) = F(s) * [|b-a|\,\text{sinc}((b-a)s)e^{-i\pi(a+b)s}]$$

More generally, we can "window" the sample with some other function $\omega(t)$:

$$f_{\text{sample}}(t) = f(t)\,\omega(t)$$

yielding

$$F_{\text{sample}}(s) = F(s) * W(s)$$

Thus, the estimated spectrum can be highly contaminated by the Fourier transform of the window, often with the effect of smoothing and distorting the spectrum due to the convolution with the window spectrum $W(s)$. This can be particularly severe in the analysis of ultrasonic waveforms in the laboratory, where often only the first 1 to $1\frac{1}{2}$ cycles are included in the window. The solution to the problem is not easy, and there is an extensive literature (e.g., Jenkins and Watts, 1968; Marple, 1987) on spectral estimation. Our advice is to be aware of the artifacts of windowing and to experiment to determine the sensitivity of the results, such as the spectral ratio or the phase velocity, to the choice of window size and shape.

Fourier transform theorems

Tables 1.1.1 and 1.1.2 summarize some useful theorems (Bracewell, 1965). If $f(x)$ has the Fourier transform $F(s)$, and $g(x)$ has the Fourier transform $G(s)$, then the Fourier

Table 1.1.1 *Fourier transform theorems.*

Theorem	x-domain		s-domain
Similarity	$f(ax)$	$\Leftrightarrow$	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$
Addition	$f(x) + g(x)$	$\Leftrightarrow$	$F(s) + G(s)$
Shift	$f(x - a)$	$\Leftrightarrow$	$e^{-i2\pi as}F(s)$
Modulation	$f(x)\cos\,\omega x$	$\Leftrightarrow$	$\frac{1}{2}F\left(s-\frac{\omega}{2\pi}\right)+\frac{1}{2}F\left(s+\frac{\omega}{2\pi}\right)$
Convolution	$f(x) * g(x)$	$\Leftrightarrow$	$F(s)\,G(s)$
Autocorrelation	$f(x) * f^*(-x)$	$\Leftrightarrow$	$\lvert F(s)\rvert^2$
Derivative	$f'(x)$	$\Leftrightarrow$	$i2\pi sF(s)$

Table 1.1.2 *Some additional theorems.*

Derivative of convolution	$\frac{d}{dx}[f(x) * g(x)] = f'(x) * g(x) = f(x) * g'(x)$
Rayleigh	$\int_{-\infty}^{\infty}\lvert f(x)\rvert^2\,dx = \int_{-\infty}^{\infty}\lvert F(s)\rvert^2\,ds$
Power	$\int_{-\infty}^{\infty} f(x)\,g^*(x)\,dx = \int_{-\infty}^{\infty} F(s)\,G^*(s)\,ds$
(f and g real)	$\int_{-\infty}^{\infty} f(x)\,g(-x)\,dx = \int_{-\infty}^{\infty} F(s)\,G(s)\,ds$

transform pairs in the x-domain and the s-domain are as shown in the tables. Table 1.1.3 lists some useful Fourier transform pairs.

1.2 The Hilbert transform and analytic signal

Synopsis

The **Hilbert transform** of $f(x)$ is defined as

$$F_{\mathrm{Hi}}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x')\,dx'}{x'-x}$$

which can be expressed as a convolution of $f(x)$ with $(-1/\pi x)$ by

$$F_{\mathrm{Hi}} = -\frac{1}{\pi x} * f(x)$$

The Fourier transform of $(-1/\pi x)$ is $(i\,\mathrm{sgn}(s))$, that is, $+i$ for positive s and $-i$ for negative s. Hence, applying the Hilbert transform keeps the Fourier amplitudes or spectrum the same but changes the phase. Under the Hilbert transform, $\sin(kx)$ is converted to $\cos(kx)$, and $\cos(kx)$ is converted to $-\sin(kx)$. Similarly, the Hilbert transforms of even functions are odd functions and vice versa.

Table 1.1.3 *Some Fourier transform pairs.*

	$\sin \pi x$	$\frac{i}{2}\left[\delta\left(s+\frac{1}{2}\right)-\delta\left(s-\frac{1}{2}\right)\right]$
	$\cos \pi x$	$\frac{1}{2}\left[\delta\left(s+\frac{1}{2}\right)+\delta\left(s-\frac{1}{2}\right)\right]$
	$\delta(x)$	1
	$\mathrm{sinc}(x)$	$\Pi(s)$
	$\mathrm{sinc}^2(x)$	$\Lambda(s)$
	$e^{-\pi x^2}$	$e^{-\pi s^2}$
	$-1/\pi x$	$i\,\mathrm{sgn}(s)$
	$\frac{x_0}{x_0^2+x^2}$	$\pi \exp(-2\pi x_0\lvert s\rvert)$
	$e^{-\lvert x\rvert}$	$\frac{2}{1+(2\pi s)^2}$
	$\lvert x\rvert^{-1/2}$	$\lvert s\rvert^{-1/2}$

The inverse of the Hilbert transform is itself the Hilbert transform with a change of sign:

$$f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F_{\mathrm{Hi}}(x')\,dx'}{x'-x}$$

or

$$f(x) = -\left(-\frac{1}{\pi x}\right) * F_{\mathrm{Hi}}$$

The **analytic signal** associated with a real function, $f(t)$, is the complex function

$$S(t) = f(t) - i\,F_{\mathrm{Hi}}(t)$$

As discussed below, the Fourier transform of $S(t)$ is zero for negative frequencies.

The **instantaneous envelope** of the analytic signal is

$$E(t) = \sqrt{f^2(t) + F_{\text{Hi}}^2(t)}$$

The **instantaneous phase** of the analytic signal is

$$\begin{aligned}\varphi(t) &= \tan^{-1}[-F_{\text{Hi}}(t)/f(t)] \\ &= \text{Im}[\ln(S(t))]\end{aligned}$$

The **instantaneous frequency** of the analytic signal is

$$\omega = \frac{d\varphi}{dt} = \text{Im}\left[\frac{d}{dt}\ln(S)\right] = \text{Im}\left(\frac{1}{S}\frac{dS}{dt}\right)$$

Claerbout (1992) has suggested that ω can be numerically more stable if the denominator is rationalized and the functions are locally smoothed, as in the following equation:

$$\bar{\omega} = \text{Im}\left[\frac{\left\langle S^*(t)\frac{dS(t)}{dt}\right\rangle}{\langle S^*(t)\,S(t)\rangle}\right]$$

where $\langle\cdot\rangle$ indicates some form of running average or smoothing.

Causality

The **impulse response**, $I(t)$, of a real physical system must be causal, that is,

$$I(t) = 0, \qquad \text{for } t < 0$$

The Fourier transform $T(f)$ of the impulse response of a causal system is sometimes called the **transfer function**:

$$T(f) = \int_{-\infty}^{\infty} I(t)\, e^{-i2\pi ft}\, dt$$

$T(f)$ must have the property that the real and imaginary parts are Hilbert transform pairs, that is, $T(f)$ will have the form

$$T(f) = G(f) + iB(f)$$

where $B(f)$ is the Hilbert transform of $G(f)$:

$$B(f) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{G(f')\,df'}{f' - f}$$

$$G(f) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{B(f')\,df'}{f' - f}$$

Similarly, if we reverse the domains, an analytic signal of the form

$$S(t) = f(t) - iF_{\mathrm{Hi}}(t)$$

must have a Fourier transform that is zero for negative frequencies. In fact, one convenient way to implement the Hilbert transform of a real function is by performing the following steps:

(1) Take the Fourier transform.
(2) Multiply the Fourier transform by zero for $f < 0$.
(3) Multiply the Fourier transform by 2 for $f > 0$.
(4) If done numerically, leave the samples at $f = 0$ and the Nyquist frequency unchanged.
(5) Take the inverse Fourier transform.

The imaginary part of the result will be the negative Hilbert transform of the real part.

1.3 Statistics and probability

Synopsis

The **sample mean**, m, of a set of n data points, x_i, is the arithmetic average of the data values:

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The **median** is the midpoint of the observed values if they are arranged in increasing order. The **sample variance**, σ^2, is the average squared difference of the observed values from the mean:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - m)^2$$

(An unbiased estimate of the **population variance** is often found by dividing the sum given above by $(n - 1)$ instead of by n.)

The **standard deviation**, σ, is the square root of the variance, while the **coefficient of variation** is σ/m. The **mean deviation**, α, is

$$\alpha = \frac{1}{n}\sum_{i=1}^{n} |x_i - m|$$

Regression

When trying to determine whether two different data variables, x and y, are related, we often estimate the **correlation coefficient**, ρ, given by (e.g., Young, 1962)

$$\rho = \frac{\frac{1}{n}\sum_{i=1}^{n} (x_i - m_x)(y_i - m_y)}{\sigma_x \sigma_y}, \qquad \text{where } |\rho| \leq 1$$

where σ_x and σ_y are the standard deviations of the two distributions and m_x and m_y are their means. The correlation coefficient gives a measure of how close the points come to falling along a straight line in a scatter plot of x versus y. $|\rho| = 1$ if the points lie perfectly along a line, and $|\rho| < 1$ if there is scatter about the line. The numerator of this expression is the **sample covariance**, C_{xy}, which is defined as

$$C_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i - m_x)(y_i - m_y)$$

It is important to remember that the correlation coefficient is a measure of the *linear* relation between x and y. If they are related in a nonlinear way, the correlation coefficient will be misleadingly small.

The simplest recipe for estimating the linear relation between two variables, x and y, is **linear regression**, in which we assume a relation of the form:

$$y = ax + b$$

The coefficients that provide the best fit to the measured values of y, in the least-squares sense, are

$$a = \rho\frac{\sigma_y}{\sigma_x}, \qquad b = m_y - am_x$$

More explicitly,

$$a = \frac{n\sum x_i y_i - (\sum x_i)(\sum y_i)}{n\sum x_i^2 - (\sum x_i)^2}, \quad \text{slope}$$

$$b = \frac{(\sum y_i)(\sum x_i^2) - (\sum x_i y_i)(\sum x_i)}{n\sum x_i^2 - (\sum x_i)^2}, \quad \text{intercept}$$

The scatter or variation of y-values around the regression line can be described by the sum of the squared errors as

$$E^2 = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

where $\hat{y}_i$ is the value predicted from the regression line. This can be expressed as a variance around the regression line as

$$\hat{\sigma}_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$$

The square of the correlation coefficient ρ is the **coefficient of determination**, often denoted by r^2, which is a measure of the regression variance relative to the total variance in the variable y, expressed as