PART 1: MATHEMATICAL PRELIMINARIES

1 Vectors and Tensors

This chapter and the next are concerned with establishing some basic properties of vectors and tensors in real spaces. The first of these is specifically concerned with vector algebra and introduces the notion of tensors; the next chapter continues the discussion of tensor algebra and introduces Gauss and Stokes's integral theorems. The discussion in both chapters is focused on laying out the algebraic methods needed in developing the concepts that follow throughout the book. It is, therefore, selective and thus far from inclusive of all vector and tensor algebra. Selected reading is recommended for additional study as it is for all subsequent chapters. Chapter 3 is an introduction to Fourier series and Fourier integrals, added to facilitate the derivation of certain elasticity solutions in later chapters of the book.

1.1 Vector Algebra

We consider three-dimensional *Euclidean vector spaces*, $\mathcal{E}$, for which to each vector such as $\mathbf{a}$ or $\mathbf{b}$ there exists a scalar formed by a *scalar product* $\mathbf{a}\cdot\mathbf{b}$ such that $\mathbf{a}\cdot\mathbf{b} =$ *a real number in* $\mathcal{R}$ and a *vector product* that is another vector such that $\mathbf{a}\times\mathbf{b} = \mathbf{c}$. Note the definitions *via* the operations of the symbols, $\cdot$ and $\times$, respectively. Connections to common geometric interpretations will be considered shortly.

With α and β being scalars, the properties of these operations are as follows

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}, \quad \forall\, \mathbf{a}, \mathbf{b} \in \mathcal{E}, \tag{1.1}$$

$$(\alpha\mathbf{a} + \beta\mathbf{b})\cdot\mathbf{c} = \alpha(\mathbf{a}\cdot\mathbf{c}) + \beta(\mathbf{b}\cdot\mathbf{c}), \quad \forall\, \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{E}, \tag{1.2}$$

$$\mathbf{a}\cdot\mathbf{a} \geq 0, \quad \text{with} \quad \mathbf{a}\cdot\mathbf{a} = 0 \quad \text{iff} \quad \mathbf{a} = \mathbf{0}, \tag{1.3}$$

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}, \tag{1.4}$$

$$(\alpha\mathbf{a} + \beta\mathbf{b})\times\mathbf{c} = \alpha(\mathbf{a}\times\mathbf{c}) + \beta(\mathbf{b}\times\mathbf{c}). \tag{1.5}$$

Also,

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = 0, \tag{1.6}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})^2. \tag{1.7}$$

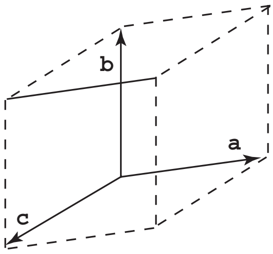

Figure 1.1. Geometric meaning of a vector triple product. The triple product is equal to the volume of the parallelepiped formed from the three defining vectors, **a**, **b**, and **c**.

The magnitude of **a** is

$$|\mathbf{a}| \equiv a = (\mathbf{a} \cdot \mathbf{a})^{1/2}. \tag{1.8}$$

Two vectors are *orthogonal* if

$$\mathbf{a} \cdot \mathbf{b} = 0. \tag{1.9}$$

From the above expressions it follows that if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then **a** and **b** are *linearly dependent*, *i.e.*, $\mathbf{a} = \alpha\mathbf{b}$ where α is any scalar.

A *triple product* is defined as

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \tag{1.10}$$

It is evident from simple geometry that the triple product is equal to the volume enclosed by the parallelepiped constructed from the vectors **a**, **b**, **c**. This is depicted in Fig. 1.1. Here, again, the listed vector properties allow us to write

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] \\ &= -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}] \\ &= -[\mathbf{c}, \mathbf{b}, \mathbf{a}] \end{aligned} \tag{1.11}$$

and

$$[\alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{c}, \mathbf{d}] = \alpha[\mathbf{a}, \mathbf{c}, \mathbf{d}] + \beta[\mathbf{b}, \mathbf{c}, \mathbf{d}]. \tag{1.12}$$

Furthermore,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = 0 \tag{1.13}$$

iff **a**, **b**, **c** are linearly dependent.

Because of the first of the properties (1.3), we can establish an *orthonormal basis* (Fig. 1.2) that we designate as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, such that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.14}$$

The δ_{ij} is referred to as the Kronecker delta. Using the basis $\{\mathbf{e}_i\}$, an arbitrary vector, say **a**, can be expressed as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \tag{1.15}$$

or

$$\mathbf{a} = a_i\mathbf{e}_i, \tag{1.16}$$

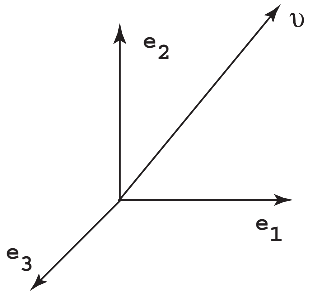

Figure 1.2. A vector υ in an orthonormal basis.

where the repeated index i implies summation, *i.e.*,

$$\mathbf{a} = a_i \mathbf{e}_i = \sum_{i=1}^{3} a_i \mathbf{e}_i . \tag{1.17}$$

We can use (1.14) to show that

$$a_i = \mathbf{a} \cdot \mathbf{e}_i = a_r \mathbf{e}_r \cdot \mathbf{e}_i = a_r \delta_{ri} . \tag{1.18}$$

The properties listed previously allow us to write

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3, \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2. \tag{1.19}$$

We note that these relations can be expressed as

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k, \tag{1.20}$$

where the *permutation tensor* is defined as

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } i, j, k \text{ are an even permutation of } 1, 2, 3, \\ -1, & \text{if } i, j, k \text{ are an odd permutation of } 1, 2, 3, \\ \ \ 0, & \text{if any of } i,\ j,\ k \text{ are the same.} \end{cases} \tag{1.21}$$

Some useful results follow. Let $\mathbf{a} = a_p \mathbf{e}_p$ and $\mathbf{b} = b_r \mathbf{e}_r$. Then,

$$\mathbf{a} \cdot \mathbf{b} = (a_p \mathbf{e}_p) \cdot (b_r \mathbf{e}_r) = a_p b_r (\mathbf{e}_p \cdot \mathbf{e}_r) = a_p b_r \delta_{pr} . \tag{1.22}$$

Thus, the scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_p b_p = a_r b_r . \tag{1.23}$$

Similarly, the vector product is

$$\mathbf{a} \times \mathbf{b} = a_p \mathbf{e}_p \times b_r \mathbf{e}_r = a_p b_r \mathbf{e}_p \times \mathbf{e}_r = a_p b_r \epsilon_{pri} \mathbf{e}_i = \epsilon_{ipr} (a_p b_r) \mathbf{e}_i . \tag{1.24}$$

Finally, the component form of the triple product,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \tag{1.25}$$

is

$$\mathbf{c} \cdot (\epsilon_{ipr} a_p b_r \mathbf{e}_i) = \epsilon_{ipr} a_p b_r c_i = \epsilon_{pri} a_p b_r c_i = \epsilon_{ijk} a_i b_j c_k . \tag{1.26}$$

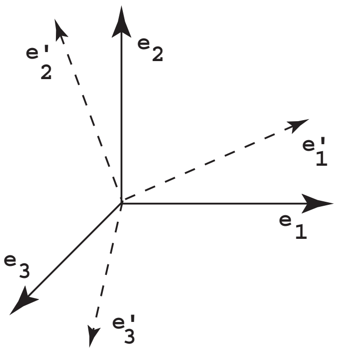

Figure 1.3. Transformation *via* rotation of basis.

1.2 Coordinate Transformation: Rotation of Axes

Let $\boldsymbol{\upsilon}$ be a vector referred to two sets of basis vectors, $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$, *i.e.*,

$$\boldsymbol{\upsilon} = \upsilon_i \mathbf{e}_i = \upsilon'_i \mathbf{e}'_i. \tag{1.27}$$

We seek to relationship of the υ_i to the υ'_i. Let the transformation between two bases (Fig. 1.3) be given by

$$\mathbf{e}'_i = \alpha_{ij} \mathbf{e}_j. \tag{1.28}$$

Then

$$\mathbf{e}'_i \cdot \mathbf{e}_j = \alpha_{is} \mathbf{e}_s \cdot \mathbf{e}_j = \alpha_{is} \delta_{sj} = \alpha_{ij}. \tag{1.29}$$

It follows that

$$\upsilon'_s = \boldsymbol{\upsilon} \cdot \mathbf{e}'_s = \boldsymbol{\upsilon} \cdot \alpha_{sp} \mathbf{e}_p = \upsilon_p \alpha_{sp} = \alpha_{sp} \upsilon_p$$

and thus

$$\upsilon'_i = \alpha_{ij} \upsilon_j. \tag{1.30}$$

For example, in the two-dimensional case, we have

$$\begin{aligned} \mathbf{e}'_1 &= \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2, \\ \mathbf{e}'_2 &= -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2, \end{aligned} \tag{1.31}$$

with the corresponding transformation matrix

$$\boldsymbol{\alpha} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{1.32}$$

Another way to describe the transformation in (1.28) is to set

$$\mathbf{e}' = \boldsymbol{\beta} \cdot \mathbf{e}. \tag{1.33}$$

A straightforward manipulation, however, shows that $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$ are related by

$$\boldsymbol{\beta} = \boldsymbol{\alpha}^T, \tag{1.34}$$

where the *transpose*, $\boldsymbol{\alpha}^T$, is defined in the sequel.

1.3 Second-Rank Tensors

A vector assigns to each direction a scalar, *viz.*, the magnitude of the vector. A second-rank tensor assigns to each vector another (unique) vector, *via* the operation

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{b}. \tag{1.35}$$

More generally,

$$\mathbf{A} \cdot (\alpha \mathbf{a} + \beta \mathbf{b}) = \alpha \mathbf{A} \cdot \mathbf{a} + \beta \mathbf{A} \cdot \mathbf{b}. \tag{1.36}$$

Second-rank tensors obey the following additional rules

$$\begin{aligned}
(\mathbf{A} + \mathbf{B}) \cdot \mathbf{a} &= \mathbf{A} \cdot \mathbf{a} + \mathbf{B} \cdot \mathbf{a}, \\
(\alpha \mathbf{A}) \cdot \mathbf{a} &= \alpha \mathbf{A} \cdot \mathbf{a}, \\
(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{a} &= \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{a}), \\
\mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A}, \\
\alpha(\mathbf{A} \cdot \mathbf{B}) &= (\alpha \mathbf{A}) \cdot \mathbf{B}, \\
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) &= \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \\
\mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}.
\end{aligned} \tag{1.37}$$

Each tensor, $\mathbf{A}$, has a unique transpose, $\mathbf{A}^T$, defined such that

$$\mathbf{a} \cdot (\mathbf{A}^T \cdot \mathbf{b}) = \mathbf{b} \cdot (\mathbf{A} \cdot \mathbf{a}). \tag{1.38}$$

Because of (1.36)–(1.38), we can write

$$(\alpha \mathbf{A} + \beta \mathbf{B})^T = \alpha \mathbf{A}^T + \beta \mathbf{B}^T, \tag{1.39}$$

and

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T. \tag{1.40}$$

1.4 Symmetric and Antisymmetric Tensors

We call the tensor $\mathbf{A}$ symmetric if $\mathbf{A} = \mathbf{A}^T$. $\mathbf{A}$ is said to be antisymmetric if $\mathbf{A} = -\mathbf{A}^T$. An arbitrary tensor, $\mathbf{A}$, can be expressed (or decomposed) in terms of its symmetric and antisymmetric parts, *via*

$$\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T), \tag{1.41}$$

where

$$\begin{aligned}
\operatorname{sym}(\mathbf{A}) &\equiv \frac{1}{2}(\mathbf{A} + \mathbf{A}^T), \\
\operatorname{skew}(\mathbf{A}) &\equiv \frac{1}{2}(\mathbf{A} - \mathbf{A}^T).
\end{aligned} \tag{1.42}$$

1.5 Prelude to Invariants of Tensors

Let $\{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$ and $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$ be two arbitrary bases of $\mathcal{E}$. Then it can be shown that

$$\begin{aligned}\chi_1 &= ([\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{g}, \mathbf{A}\cdot\mathbf{h}])\,/[\mathbf{f}, \mathbf{g}, \mathbf{h}]\\ &= ([\mathbf{A}\cdot\mathbf{l}, \mathbf{m}, \mathbf{n}] + [\mathbf{l}, \mathbf{A}\cdot\mathbf{m}, \mathbf{n}] + [\mathbf{l}, \mathbf{m}, \mathbf{A}\cdot\mathbf{n}])\,/[\mathbf{l}, \mathbf{m}, \mathbf{n}],\end{aligned}$$

$$\begin{aligned}\chi_2 &= ([\mathbf{A}\cdot\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{A}\cdot\mathbf{h}] + [\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{A}\cdot\mathbf{h}])\,/[\mathbf{f}, \mathbf{g}, \mathbf{h}]\\ &= ([\mathbf{A}\cdot\mathbf{l}, \mathbf{A}\cdot\mathbf{m}, \mathbf{n}] + [\mathbf{l}, \mathbf{A}\cdot\mathbf{m}, \mathbf{A}\cdot\mathbf{n}] + [\mathbf{A}\cdot\mathbf{l}, \mathbf{m}, \mathbf{A}\cdot\mathbf{n}])\,/[\mathbf{l}, \mathbf{m}, \mathbf{n}],\end{aligned}$$

$$\chi_3 = [\mathbf{A}\cdot\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{A}\cdot\mathbf{h}]/[\mathbf{f}, \mathbf{g}, \mathbf{h}] = [\mathbf{A}\cdot\mathbf{l}, \mathbf{A}\cdot\mathbf{m}, \mathbf{A}\cdot\mathbf{n}]/[\mathbf{l}, \mathbf{m}, \mathbf{n}].$$

In proof of the first of the above, consider the first part of the *lhs*,

$$\begin{aligned}[\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{h}] &= [\mathbf{A}\cdot(f_p\mathbf{e}_p), g_q\mathbf{e}_q, h_r\mathbf{e}_r]\\ &= [f_p(\mathbf{A}\cdot\mathbf{e}_p), g_q\mathbf{e}_q, h_r\mathbf{e}_r] = f_p g_q h_r[\mathbf{A}\cdot\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r].\end{aligned} \tag{1.43}$$

Thus the entire expression for χ_1 becomes

$$\chi_1 = \frac{f_p g_q h_r}{[\mathbf{f}, \mathbf{g}, \mathbf{h}]}\left([\mathbf{A}\cdot\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r] + [\mathbf{e}_p, \mathbf{A}\cdot\mathbf{e}_q, \mathbf{e}_r] + [\mathbf{e}_p, \mathbf{e}_q, \mathbf{A}\cdot\mathbf{e}_r]\right). \tag{1.44}$$

The term in $(\ldots)$ remains unchanged if p, q, r undergo an even permutation of $1, 2, 3$; it reverses sign if p, q, r undergo an odd permutation, and is equal to 0 if any of p, q, r are made equal. Thus set $p = 1,\ q = 2,\ r = 3$, and multiply the result by ϵ_{pqr} to take care of the changes in sign or the null results just described. The full expression becomes

$$\begin{aligned}&\frac{(f_p g_q h_r)\epsilon_{pqr}}{[\mathbf{f}, \mathbf{g}, \mathbf{h}]}\left([\mathbf{A}\cdot\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{A}\cdot\mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \mathbf{A}\cdot\mathbf{e}_3]\right)\\ &= [\mathbf{A}\cdot\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{A}\cdot\mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \mathbf{A}\cdot\mathbf{e}_3].\end{aligned} \tag{1.45}$$

Since $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = +1$, the quantity

$$([\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{g}, \mathbf{A}\cdot\mathbf{h}])\,/[\mathbf{f}, \mathbf{g}, \mathbf{h}] \tag{1.46}$$

is invariant to changes of the basis $\{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$.

Given the validity of the expressions for χ_1, χ_2, and χ_3, we thereby discover three invariants of the tensor $\mathbf{A}$, *viz.*,

$$\begin{aligned}I_A &= ([\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{g}, \mathbf{A}\cdot\mathbf{h}])\,/[\mathbf{f}, \mathbf{g}, \mathbf{h}],\\ II_A &= ([\mathbf{A}\cdot\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{h}] + [\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{A}\cdot\mathbf{h}] + [\mathbf{A}\cdot\mathbf{f}, \mathbf{g}, \mathbf{A}\cdot\mathbf{h}])\,/[\mathbf{f}, \mathbf{g}, \mathbf{h}],\\ III_A &= [\mathbf{A}\cdot\mathbf{f}, \mathbf{A}\cdot\mathbf{g}, \mathbf{A}\cdot\mathbf{h}]/[\mathbf{f}, \mathbf{g}, \mathbf{h}].\end{aligned} \tag{1.47}$$

The commonly held descriptors for two of these are

$$I_A = \text{trace of } \mathbf{A} = \operatorname{tr}(\mathbf{A}),$$

$$III_A = \text{determinant of } \mathbf{A} = \det(\mathbf{A}) = |\mathbf{A}|.$$

1.6 Inverse of a Tensor

If $|\mathbf{A}| \neq 0$, $\mathbf{A}$ has an inverse, $\mathbf{A}^{-1}$, such that

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}, \tag{1.48}$$

where $\mathbf{I}$, *the identity tensor*, is defined *via* the relations

$$\mathbf{a} = \mathbf{I} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{I}. \tag{1.49}$$

Useful relations that follow from the above are

$$\begin{aligned} |\alpha \mathbf{A}| &= \alpha^3 |\mathbf{A}|, \\ |\mathbf{A} \cdot \mathbf{B}| &= |\mathbf{A}||\mathbf{B}|. \end{aligned} \tag{1.50}$$

Thus, it follows that

$$\begin{aligned} |\mathbf{A} \cdot \mathbf{A}^{-1}| &= |\mathbf{I}| = 1 = |\mathbf{A}||\mathbf{A}^{-1}|, \\ |\mathbf{A}^{-1}| &= \frac{1}{|\mathbf{A}|} = |\mathbf{A}|^{-1}. \end{aligned} \tag{1.51}$$

1.7 Additional Proofs

We deferred formal proofs of several lemmas until now in the interest of presentation. We provide the proofs at this time.

LEMMA 1.1: *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are two vectors,* $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ *iff* $\mathbf{a}$ *and* $\mathbf{b}$ *are linearly dependent.*

Proof: If $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent then there is a scalar such that $\mathbf{b} = \alpha \mathbf{a}$. In this case, if we express the vector product $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ in component form, we find that $c_i = \epsilon_{ijk} a_j \alpha a_k = \alpha \epsilon_{ijk} a_j a_k$. But the summations over the indices j and k will produce pairs of multiples of $a_\beta a_\gamma$, and then again $a_\gamma a_\beta$, for which the permutator tensor alternates algebraic sign, thus causing such pairs to cancel. Thus, in this case $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Conversely, if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, we find from (1.3) to (1.8) that $\mathbf{a} \times \mathbf{b} = \pm |\mathbf{a}||\mathbf{b}|$. If the plus signs holds, we have from the second of (1.3)

$$(|\mathbf{b}|\mathbf{a} - |\mathbf{a}|\mathbf{b}) \cdot (|\mathbf{b}|\mathbf{a} - |\mathbf{a}|\mathbf{b}) = 2|\mathbf{a}|^2|\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\mathbf{a} \cdot \mathbf{b} = 0. \tag{1.52}$$

Because of the third property in (1.3) this means that $|\mathbf{b}|\mathbf{a} = |\mathbf{a}|\mathbf{b}$. When the minus sign holds, we find that $|\mathbf{b}|\mathbf{a} = -|\mathbf{a}|\mathbf{b}$. In either case this leads to the conclusion that $\mathbf{b} = \alpha \mathbf{a}$.

Next we examine the relations defining properties of the triple product when pairs of the vectors are interchanged. Use (1.26) to calculate the triple product. This yields $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon_{ijk} a_i b_j c_k$. Next imagine interchanging, say $\mathbf{a}$ with $\mathbf{b}$; we obtain $[\mathbf{b}, \mathbf{a}, \mathbf{c}] = \epsilon_{ijk} b_i a_j c_k = \epsilon_{ijk} a_j b_i c_k = -\epsilon_{jik} a_j b_i c_k = -\epsilon_{ijk} a_i b_j c_k$, where the last term involved merely a reassignment of summation indices. Thus $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}]$. Proceeding this way all members of (1.11) are generated.

We now examine the triple product property expressed in (1.12).

LEMMA 1.2: *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, *and* $\mathbf{d}$ *are arbitrary vectors, and* α *and* β *arbitrary scalars, then*

$$[\alpha \mathbf{a} + \beta \mathbf{b}, \mathbf{c}, \mathbf{d}] = \alpha [\mathbf{a}, \mathbf{c}, \mathbf{d}] + \beta [\mathbf{b}, \mathbf{c}, \mathbf{d}], \quad \forall\, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{E}, \quad \alpha, \beta \in \mathcal{R}. \tag{1.53}$$

Proof: Begin with the property of scalar products between vectors expressed in (1.2) and replace $\mathbf{c}$ with $\mathbf{c} \times \mathbf{d}$. Then,

$$(\alpha \mathbf{a} + \beta \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \alpha \mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}) + \beta \mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}) = \alpha[\mathbf{a}, \mathbf{c}, \mathbf{d}] + \beta[\mathbf{b}, \mathbf{c}, \mathbf{d}]. \tag{1.54}$$

Of course, the first term in the above is the triple product expressed on the *lhs* of the lemma.

1.8 Additional Lemmas for Vectors

LEMMA 1.3: *If*

$$\boldsymbol{v} = \alpha \mathbf{a} + \beta \boldsymbol{b} + \gamma \boldsymbol{c}, \tag{1.55}$$

where $\mathbf{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{v}$ *are all vectors, then*

$$\alpha = \frac{\epsilon_{ijk} v_i b_j c_k}{\epsilon_{pqr} a_p b_q c_r}, \quad \beta = \frac{\epsilon_{ijk} a_i v_j c_k}{\epsilon_{pqr} a_p b_q c_r}, \quad \gamma = \frac{\epsilon_{ijk} a_i b_j v_k}{\epsilon_{pqr} a_p b_q c_r}. \tag{1.56}$$

Proof: The three relations that express the connections are

$$\begin{aligned} v_1 &= \alpha a_1 + \beta b_1 + \gamma c_1, \\ v_2 &= \alpha a_2 + \beta b_2 + \gamma c_2, \\ v_3 &= \alpha a_3 + \beta b_3 + \gamma c_3. \end{aligned} \tag{1.57}$$

By Cramer's rule

$$\alpha = \frac{\begin{vmatrix} v_1 & b_1 & c_1 \\ v_2 & b_2 & c_2 \\ v_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{1.58}$$

Thus, the lemma is proved once the two determinants are expressed using the permutation tensor.

LEMMA 1.4: *Given a vector* $\mathbf{a}$, *then for arbitrary vector* $\mathbf{x}$,

$$\mathbf{a} \times \mathbf{x} = \mathbf{a} \;\text{ iff }\; \mathbf{a} = \mathbf{0}. \tag{1.59}$$

Proof: Express the i^{th} component of $\mathbf{a} \times \mathbf{x}$ as

$$\epsilon_{ijk} a_j x_k, \tag{1.60}$$

and then form the product $\mathbf{a} \cdot \mathbf{a}$ to obtain

$$\epsilon_{ijk} a_j x_k \epsilon_{irs} a_r x_s = (\delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}) a_j x_k a_r a_s = a^2 x^2 - (\mathbf{a} \cdot \mathbf{x})^2 = a^2.$$

The expression just generated is zero as may be seen, for example, by letting $\mathbf{x}$ be equal to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively. Note that the third equation of (1.70) below has been used.

LEMMA 1.5: *Suppose that for any vector* $\mathbf{p}$, $\mathbf{p}\cdot\mathbf{q}=\mathbf{p}\cdot\mathbf{t}$, *then we have*

$$\mathbf{p}\cdot\mathbf{q}=\mathbf{p}\cdot\mathbf{t} \quad\Rightarrow\quad \mathbf{q}=\mathbf{t}. \tag{1.61}$$

Proof: The relation $\mathbf{p}\cdot\mathbf{q}=\mathbf{p}\cdot\mathbf{t}$ can be rewritten as

$$\mathbf{p}\cdot(\mathbf{q}-\mathbf{t})=p_1(q_1-t_1)+p_2(q_2-t_2)+p_3(q_3-t_3)=0. \tag{1.62}$$

As in the previous lemma, letting $\mathbf{p}$ be systematically equal to $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ shows that $p_1=p_2=p_3=0$.

We reexamine now the operation of the cross product between vectors to develop two additional lemmas of interest.

LEMMA 1.6: *Given the vectors* $\mathbf{p},\mathbf{q},\mathbf{r}$ *we have*

$$\mathbf{p}\times(\mathbf{q}\times\mathbf{r})=\mathbf{q}(\mathbf{p}\cdot\mathbf{r})-\mathbf{r}(\mathbf{p}\cdot\mathbf{q}). \tag{1.63}$$

Proof: The proof is most readily done by expressing the above in component form, *i.e.*,

$$\epsilon_{rsi}p_s\epsilon_{ijk}q_jr_k=\epsilon_{rsi}\epsilon_{ijk}p_sq_jr_k=\epsilon_{irs}\epsilon_{ijk}p_sq_jr_k. \tag{1.64}$$

Use the identity given by the third equation of (1.70) and write

$$\epsilon_{irs}\epsilon_{ijk}p_sq_jr_k=q_r(p_sr_s)-r_r(p_sq_s) \tag{1.65}$$

to complete the proof.

A simple extension of the last lemma is that

$$(\mathbf{p}\times\mathbf{q})\times\mathbf{r}=\mathbf{q}(\mathbf{p}\cdot\mathbf{r})-\mathbf{p}(\mathbf{r}\cdot\mathbf{q}). \tag{1.66}$$

The proof is left as an exercise.

1.9 Coordinate Transformation of Tensors

Consider coordinate transformations prescribed by (1.28). A tensor $\mathbf{A}$ can be written alternatively as

$$\mathbf{A}=A_{ij}\mathbf{e}_i\mathbf{e}_j=A'_{ij}\mathbf{e}'_i\mathbf{e}'_j=A'_{ij}\alpha_{ir}\mathbf{e}_r\alpha_{js}\mathbf{e}_s. \tag{1.67}$$

Since $\mathbf{e}'_p\cdot\mathbf{A}\cdot\mathbf{e}'_q=A'_{pq}$, performing this operation on (1.67) gives

$$A'_{pq}=\mathbf{e}'_p\cdot A_{ij}\mathbf{e}_i\mathbf{e}_j\cdot\mathbf{e}'_q=\alpha_{pi}\alpha_{qj}A_{ij}. \tag{1.68}$$

Transformation of higher order tensors can be handled in an identical manner.

1.10 Some Identities with Indices

The following identities involving the Kronecker delta are useful and are easily verified by direct expansion

$$\begin{aligned} \delta_{ii} &= 3, \\ \delta_{ij}\delta_{ij} &= 3, \\ \delta_{ij}\delta_{ik}\delta_{jk} &= 3, \\ \delta_{ij}\delta_{jk} &= \delta_{ik}, \\ \delta_{ij}A_{ik} &= A_{jk}. \end{aligned} \tag{1.69}$$

Useful identities involving the permutation tensor are

$$\begin{aligned} \epsilon_{ijk}\epsilon_{kpq} &= \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \\ \epsilon_{pqs}\epsilon_{sqr} &= -2\delta_{pr}, \\ \epsilon_{ijk}\epsilon_{ijk} &= 6. \end{aligned} \tag{1.70}$$

1.11 Tensor Product

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors; then there is a tensor $\mathbf{B} = \mathbf{uv}$ defined *via* its action on an arbitrary vector $\mathbf{a}$, such that

$$(\mathbf{uv}) \cdot \mathbf{a} = (\mathbf{v} \cdot \mathbf{a})\mathbf{u}. \tag{1.71}$$

Note that there is the commutative property that follows, *viz.*,

$$\begin{aligned} (\alpha\mathbf{u} + \beta\mathbf{v})\mathbf{w} &= \alpha\mathbf{uw} + \beta\mathbf{vw}, \\ \mathbf{u}(\alpha\mathbf{v} + \beta\mathbf{w}) &= \alpha\mathbf{uv} + \beta\mathbf{uw}. \end{aligned} \tag{1.72}$$

By the definition of the transpose as given previously, we also have

$$(\mathbf{uv})^T = \mathbf{vu}. \tag{1.73}$$

The identity tensor, $\mathbf{I}$, can be expressed as

$$\mathbf{I} = \mathbf{e}_p\mathbf{e}_p, \tag{1.74}$$

if $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are orthonormal. Indeed,

$$(\mathbf{e}_p\mathbf{e}_p) \cdot \mathbf{a} = (\mathbf{a} \cdot \mathbf{e}_p)\mathbf{e}_p = a_p\mathbf{e}_p = \mathbf{a} = \mathbf{I} \cdot \mathbf{a}.$$

LEMMA 1.7: *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are arbitrary vectors, then*

$$|\mathbf{uv}| = 0, \quad \text{and} \quad \operatorname{tr}(\mathbf{uv}) = \mathbf{u} \cdot \mathbf{v}. \tag{1.75}$$

Proof: Replace $\mathbf{A}$ in (1.47) with $\mathbf{uv}$, and use $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ as a basis of $\mathcal{E}$. Then the third equation from (1.47) becomes

$$[(\mathbf{uv}) \cdot \mathbf{a}, (\mathbf{uv}) \cdot \mathbf{b}, (\mathbf{uv}) \cdot \mathbf{c}] = III_{uv}[\mathbf{a}, \mathbf{b}, \mathbf{c}],$$