Lectures on the Combinatorics of Free Probability

Alexandru Nica
University of Waterloo, Ontario

Roland Speicher
Queen’s University, Ontario
Dedicated to Anisoara and Betina
# Contents

Introduction xiii

Part 1. Basic concepts 1

Lecture 1. Non-commutative probability spaces and distributions 3
   - Non-commutative probability spaces 3
   - *-distributions (case of normal elements) 7
   - *-distributions (general case) 13
   - Exercises 15

Lecture 2. A case study of non-normal distribution 19
   - Description of the example 19
   - Dyck paths 22
   - The distribution of $a + a^*$ 26
   - Using the Cauchy transform 30
   - Exercises 33

Lecture 3. $C^*$-probability spaces 35
   - Functional calculus in a $C^*$-algebra 35
   - $C^*$-probability spaces 39
   - *-distribution, norm and spectrum for a normal element 43
   - Exercises 46

Lecture 4. Non-commutative joint distributions 49
   - Joint distributions 49
   - Joint *-distributions 53
   - Joint *-distributions and isomorphism 55
   - Exercises 59

Lecture 5. Definition and basic properties of free independence 63
   - The classical situation: tensor independence 63
   - Definition of free independence 64
   - The example of a free product of groups 66
   - Free independence and joint moments 69
   - Some basic properties of free independence 71
   - Are there other universal product constructions? 75
## CONTENTS

<table>
<thead>
<tr>
<th>Exercises</th>
<th>78</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lecture 6. Free product of $\ast$-probability spaces</td>
<td>81</td>
</tr>
<tr>
<td>Free product of unital algebras</td>
<td>81</td>
</tr>
<tr>
<td>Free product of non-commutative probability spaces</td>
<td>84</td>
</tr>
<tr>
<td>Free product of $\ast$-probability spaces</td>
<td>86</td>
</tr>
<tr>
<td>Exercises</td>
<td>92</td>
</tr>
<tr>
<td>Lecture 7. Free product of $\mathcal{C}^\ast$–probability spaces</td>
<td>95</td>
</tr>
<tr>
<td>The GNS representation</td>
<td>95</td>
</tr>
<tr>
<td>Free product of $\mathcal{C}^\ast$-probability spaces</td>
<td>99</td>
</tr>
<tr>
<td>Example: semicircular systems and the full Fock space</td>
<td>102</td>
</tr>
<tr>
<td>Exercises</td>
<td>109</td>
</tr>
</tbody>
</table>

### Part 2. Cumulants

<table>
<thead>
<tr>
<th>Exercises</th>
<th>113</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lecture 8. Motivation: free central limit theorem</td>
<td>115</td>
</tr>
<tr>
<td>Convergence in distribution</td>
<td>115</td>
</tr>
<tr>
<td>General central limit theorem</td>
<td>117</td>
</tr>
<tr>
<td>Classical central limit theorem</td>
<td>120</td>
</tr>
<tr>
<td>Free central limit theorem</td>
<td>121</td>
</tr>
<tr>
<td>The multi-dimensional case</td>
<td>125</td>
</tr>
<tr>
<td>Conclusion and outlook</td>
<td>131</td>
</tr>
<tr>
<td>Exercises</td>
<td>132</td>
</tr>
<tr>
<td>Lecture 9. Basic combinatorics I: non-crossing partitions</td>
<td>135</td>
</tr>
<tr>
<td>Non-crossing partitions of an ordered set</td>
<td>135</td>
</tr>
<tr>
<td>The lattice structure of $NC(n)$</td>
<td>144</td>
</tr>
<tr>
<td>The factorization of intervals in $NC$</td>
<td>148</td>
</tr>
<tr>
<td>Exercises</td>
<td>153</td>
</tr>
<tr>
<td>Lecture 10. Basic combinatorics II: Möbius inversion</td>
<td>155</td>
</tr>
<tr>
<td>Convolution in the framework of a poset</td>
<td>155</td>
</tr>
<tr>
<td>Möbius inversion in a lattice</td>
<td>160</td>
</tr>
<tr>
<td>The Möbius function of $NC$</td>
<td>162</td>
</tr>
<tr>
<td>Multiplicative functions on $NC$</td>
<td>164</td>
</tr>
<tr>
<td>Functional equation for convolution with $\mu_n$</td>
<td>168</td>
</tr>
<tr>
<td>Exercises</td>
<td>171</td>
</tr>
<tr>
<td>Lecture 11. Free cumulants: definition and basic properties</td>
<td>173</td>
</tr>
<tr>
<td>Multiplicative functionals on $NC$</td>
<td>173</td>
</tr>
<tr>
<td>Definition of free cumulants</td>
<td>175</td>
</tr>
<tr>
<td>Products as arguments</td>
<td>178</td>
</tr>
<tr>
<td>Free independence and free cumulants</td>
<td>182</td>
</tr>
</tbody>
</table>
## CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cumulants of random variables</td>
<td>185</td>
<td></td>
</tr>
<tr>
<td>Example: semicircular and circular elements</td>
<td>187</td>
<td></td>
</tr>
<tr>
<td>Even elements</td>
<td>188</td>
<td></td>
</tr>
<tr>
<td>Appendix: classical cumulants</td>
<td>190</td>
<td></td>
</tr>
<tr>
<td>Exercises</td>
<td>193</td>
<td></td>
</tr>
</tbody>
</table>

Lecture 12. Sums of free random variables
- Free convolution | 195 |
- Analytic calculation of free convolution | 200 |
- Proof of the free central limit theorem via $R$-transform | 202 |
- Free Poisson distribution | 203 |
- Compound free Poisson distribution | 206 |
- Exercises | 208 |

Lecture 13. More about limit theorems and infinitely divisible distributions
- Limit theorem for triangular arrays | 211 |
- Cumulants of operators on Fock space | 214 |
- Infinitely divisible distributions | 215 |
- Conditionally positive definite sequences | 216 |
- Characterization of infinitely divisible distributions | 220 |
- Exercises | 221 |

Lecture 14. Products of free random variables
- Multiplicative free convolution | 223 |
- Combinatorial description of free multiplication | 225 |
- Compression by a free projection | 228 |
- Convolution semigroups $(\mu_t)_{t \geq 1}$ | 231 |
- Compression by a free family of matrix units | 233 |
- Exercises | 236 |

Lecture 15. $R$-diagonal elements
- Motivation: cumulants of Haar unitary elements | 237 |
- Definition of $R$-diagonal elements | 240 |
- Special realizations of tracial $R$-diagonal elements | 245 |
- Product of two free even elements | 249 |
- The free anti-commutator of even elements | 251 |
- Powers of $R$-diagonal elements | 253 |
- Exercises | 254 |

Part 3. Transforms and models

Lecture 16. The $R$-transform
- The multi-variable $R$-transform | 259 |
The functional equation for the $R$-transform  
More about the one-dimensional case  
Exercises  

Lecture 17. The operation of boxed convolution  
The definition of boxed convolution, and its motivation  
Basic properties of boxed convolution  
Radial series  
The Möbius series and its use  
Exercises  

Lecture 18. More on the one-dimensional boxed convolution  
Relation to multiplicative functions on $NC$  
The $S$-transform  
Exercises  

Lecture 19. The free commutator  
Free commutators of even elements  
Free commutators in the general case  
The cancelation phenomenon  
Exercises  

Lecture 20. $R$-cyclic matrices  
Definition and examples of $R$-cyclic matrices  
The convolution formula for an $R$-cyclic matrix  
$R$-cyclic families of matrices  
Applications of the convolution formula  
Exercises  

Lecture 21. The full Fock space model for the $R$-transform  
Description of the Fock space model  
An application: revisiting free compressions  
Exercises  

Lecture 22. Gaussian random matrices  
Moments of Gaussian random variables  
Random matrices in general  
Selfadjoint Gaussian random matrices and genus expansion  
Asymptotic free independence for several independent Gaussian random matrices  
Asymptotic free independence between Gaussian random matrices and constant matrices  

Lecture 23. Unitary random matrices  
Haar unitary random matrices
The length function on permutations 381
Asymptotic freeness for Haar unitary random matrices 384
Asymptotic freeness between randomly rotated constant matrices 385
Embedding of non-crossing partitions into permutations 390
Exercises 393
Notes and comments 395
References 405
Index 411
Introduction

Free probability theory is a quite recent theory, bringing together many different fields of mathematics, for example operator algebras, random matrices, combinatorics, or representation theory of symmetric groups. So it has a lot to offer to various mathematical communities, and interest in free probability has steadily increased in recent years.

However, this diversity of the field also has the consequence that it is considered hard to access for a beginner. Most of the literature on free probability consists of a mixture of operator algebraic and probabilistic notions and arguments, interwoven with random matrices and combinatorics.

Whereas more advanced operator algebraic or probabilistic expertise might indeed be necessary for a deeper appreciation of special applications in the respective fields, the basic core of the theory, however, can be mostly freed from this and it is possible to give a fairly elementary introduction to the main notions, ideas and problems of free probability theory. The present lectures are intended to provide such an introduction.

Our main emphasis will be on the combinatorial side of free probability. Even when stripped from analytical structure, the main features of free independence are still present; moreover, even on this more combinatorial level it is important to organize all relevant information about the considered variables in the right way. Anyone who has tried to perform computations of joint distributions for non-commuting variables will probably agree that they tend to be horribly messy if done in a naive way. One of the main goals of the book is to show how such computations can be vastly simplified by appropriately relying on a suitable combinatorial structure – the lattices of non-crossing partitions. The combinatorial development starts from the standard theory of Möbius inversion on non-crossing partitions, but has its own specific flavor – one arrives to a theory of free or non-crossing cumulants or, in an alternative approach, one talks about $R$-transforms for non-commutative random variables.
INTRODUCTION

While writing this book, there were two kinds of readers that we had primarily in mind:

(a) a reader with background in operator algebras or probability who wants to see the more advanced “tools of the trade” on the combinatorial side of free probability;

(b) a reader with background from algebraic combinatorics who wants to get acquainted with a field (and a possible source of interesting problems) where non-trivial combinatorial tools are used.

We wrote our lectures by trying to accommodate readers from both these categories. The result is a fairly elementary exposition, which should be accessible to a beginning graduate student or even to a strong senior undergraduate student.

Free probability also has applications outside of mathematics, in particular in electrical engineering. Our exposition should also be useful for readers with engineering background, who have seen the use of $R$- or $S$-transform techniques in applications, for example in wireless communications, and who want to learn more about the underlying theory.

We emphasize that the presentation style used throughout the book is a detailed one, making the material largely self-contained, and only rarely requiring that other textbooks or research papers are consulted. The basic units of this book are called “lectures.” They were written following the idea that the material contained in one of them should be suitable for being presented in one class of a first-year graduate course. (We have in mind a class of 90 minutes, where the instructor presents the essential points of the lecture, and leaves a number of things for individual study.)

While the emphasis is on combinatorial aspects, we still felt that we must give an introduction of how the general framework of free probability comes about. Also, we felt that the flavor of the theory will be better conveyed if we show, with moderation and within a self-contained exposition, how analytical arguments can be interwoven with the combinatorial ones. However, it should be understood that in the analytical respects, this book is only an appetizer and an invitation to further reading. In particular, the analytical framework used for illustrations is exclusively that of a $C^*$-probability space. The reader should be aware that some of the most significant applications of free probability to operator algebras take place in the more elaborate framework of $W^*$-probability spaces; but going to $W^*$-structures (or in other words, to von Neumann algebra theory) did not seem possible within
the detailed, self-contained style of the book, and within the given page limits.

A consequence of the frugality of the analytic aspects covered by the book is that we do not discuss free entropy and free Fisher information, and how free cumulants can be used in some cases to perform free information calculations. Free entropy is currently one of the main directions of development in free probability; for an overview of the topic see the recent survey by Voiculescu [85].

Coming to things that are not covered by the book we must also say, with regret, that we only consider free independence over the complex field. The combinatorial ideas of free probability have a far-reaching extension to the situation when free independence is considered over an algebra $B$ (instead of just $\mathbb{C}$) – the reader interested in this direction is referred to the memoir [73].

References to the literature are not made in the body of the lectures, but are collected in the “Notes and comments” section at the end of the book. The literature on free probability is growing at an explosive rate, and, with due apologies, we felt it is beyond our limits to even try to provide an exhaustive bibliography. We have followed the line of only citing the research work which is presented in the lectures, or is very directly connected to it. For a more complete image of work in this field, the reader can consult the survey papers indicated at the beginning of the “Notes and comments” section.

So, to summarize, from one point of view this is a research monograph, presenting the current state of the combinatorial facet of free probability. At the same time it is an introduction to the field – one which is, we hope, friendly and self-contained. Finally, the book is written with the specific purpose of being used for teaching a course. We hope this will be a contribution towards making free probability appear more often as a topic for a graduate course, and we look forward to hearing from other people how following these lectures has worked for them.

Finally we would like to mention that the idea of writing this book came from a sequence of lectures which we gave at the Henri Poincaré Institute in Paris, during a special semester on free probability and operator spaces hosted by the institute in Fall 1999. Time has flown quickly since then, but we hope it is not too late to thank the Poincaré Institute, and particularly the organizers of that special semester – Philippe Biane, Gilles Pisier, and Dan Voiculescu – for the great environment they offered us, and for the opportunity of getting started on this project.