The new discipline of nonlinear dynamics, or chaos, has developed explosively in all areas of physics over the last two decades. *Chaos and Complexity in Astrophysics* provides a primer on nonlinear dynamics, and gives researchers and graduate students of astrophysics the investigative tools they need to explore chaotic and complex phenomena. Comprehensive mathematical concepts and techniques are introduced at an appropriate level in the first part of the book, before being applied to stellar, interstellar, galactic and large scale complex phenomena in the Universe. The book demonstrates the application of ideas such as strange attractors, Poincaré sections, fractals, bifurcations, complex spatial patterns, and so on, to specific astrophysical problems. This self-contained text will appeal to a broad audience of astrophysicists and astronomers who wish to learn, and then apply, modern dynamical approaches to the problems they are working on.

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To Dani, Tamar and Judy

Read not to contradict and confute,
nor to believe and take for granted,
nor to find talk and discourse,
but to weigh and consider.
Francis Bacon, Of Studies.
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Preface

The last thing one discovers in composing a work
is what to put first.
Blaise Pascal, Pensées no. 19.

In the last two decades or so the astrophysical community – students, teachers and researchers alike – have become aware of a new kind of activity in physics. Some researchers, science historians and philosophers have gone as far as calling it a ‘new science’ or ‘new physics’, while others see it as a mere natural extension of ‘old’ classical mechanics and fluid dynamics. In any case, the subject, variously referred to as dynamical systems theory, nonlinear dynamics or simply chaos, has undergone an explosive development, causing a lot of excitement in the scientific community and even in the general public. The discoveries look fundamental and there is hope that we will quite soon gain new and basic scientific understanding of the most complex aspects of nature.

The most striking quality of this modern approach to dynamical systems theory is, in my view, its extremely diverse range of applicability. Mechanics, fluid dynamics, chemical kinetics, electronic circuits, biology and even economics, as well as astrophysics, are among the subjects in which chaotic behaviour occurs. At the heart of the theory lies the quest for the universal and the generic, from which an understanding of complicated and seemingly heterogeneous phenomena can emerge. The ideas of bifurcations, strange attractors, fractal sets and so on, seem to provide the tools for such an unexpected conceptual unification.

My own experience in discussing the subject with astrophysicists suggests that they and their students would like to know more about the new developments in nonlinear dynamics. There is growing interest in the general subject as well as its possible relevance to specific fields of research or study in astrophysics.

The literature on chaos has grown enormously over the years. Books at all levels abound, from the popular to the rigorously mathematical; some of them excellent, some less so. It is not easy for an astrophysicist to pick the right book and to
learn the relevant topics from it. Astrophysical applications are dispersed through various journals and a handful of conference proceedings. There seems to be a need for a book presenting the relevant general material and coherently suggesting its possible use in astrophysics. The purpose of this book is to answer this need by providing a useful (one hopes) source of information (and perhaps new ideas) for the entire astrophysical and astronomical community, from senior undergraduates right up to established researchers.

The book is divided into two parts of quite different nature (and size). The first is devoted to a quite comprehensive description of the basic notions in dynamical systems theory and of a few more advanced theoretical topics, that may be important in astrophysics. This part may serve as a self-contained text book for an introductory course in nonlinear dynamics, especially for senior undergraduate students. The second part of the book reviews astrophysical topics, in which work in the context of dynamical systems has already been done or is in progress. Its different chapters could also be used in senior undergraduate and graduate courses in astrophysics, where nonlinear stellar pulsation, convection and hydrodynamical turbulence, the complexity of the interstellar medium, galactic dynamics or large scale structure are discussed. These topics may be taught in a broader than usual way by including, in addition to the standard material, ideas based on dynamical system theory.

Throughout the text only a basic knowledge of ordinary differential equations plus the notion of a partial differential equation are required as the mathematical background. In physics, familiarity with classical analytical mechanics and some fluid dynamics are assumed. Most of the astrophysical applications are explained in a self-contained manner, and assume only quite basic knowledge in astrophysics, which is usually acquired in a typical introductory course.

The first part of the book opens with a short introduction following which, in Chapter 2, a few dynamical systems are presented as typical examples. I have decided to dress some of these celebrated paradigms of chaos (the logistic map, nonlinear oscillators, Hamiltonian non-integrability, transition to turbulence in convection, reaction-diffusion equations) in astrophysical attire. The resulting ‘toy models’ capture the essentials and thus can serve as a demonstration of the key concepts in nonlinear and chaotic dynamics.

In Chapter 3 these concepts are systematically explored and formulated in rather abstract mathematical language. This chapter is the heart of the first part as it introduces the generic characterisation of chaotic behaviour. In the following chapter strange attractors are defined and discussed and the theoretical tools of Chapter 3 are used in the study of chaotic behaviour and the various types of transitions to it. In Chapter 5 the theoretical diagnostic tools are used in the analysis of time-series data. This type of output is expected from real experiments in the lab, astronomical
observations and numerical experiments. Chapters 6 and 7 introduce the basic concepts of Hamiltonian chaos and spatio-temporal pattern theory, respectively. The last four chapters of the first part are particularly valuable for the understanding of the astrophysical applications discussed in the second part of the book.

In the second part of the book we start by discussing several astrophysical Hamiltonian systems. Some recent work in planetary science, the question of stability of the Solar System, binary system dynamics and topics in galactic dynamics are among the subjects discussed. Next, in Chapter 10, we deal with variable astrophysical sources and start with the question of assessing the value of temporally variable astronomical signals as chaos indicators. This is critically examined in view of modern methods, some of which were introduced in Chapter 5. Several examples of theoretical models of irregular astronomical sources, based on nonlinear dynamical systems, follow. Nonlinear stellar pulsation, in which research in this context has been the most fruitful and significant, is given most attention. Chapter 11 describes some attempts at modelling the complexity of extended astrophysical media using dynamical systems and pattern theory, and we include a short discussion of the still controversial subject of a possibly fractal Universe. Finally, in Chapter 12 we discuss a number of fluid dynamical processes, whose understanding is essential to theoretical models of many important astrophysical objects. This chapter is different from all the other chapters in the second part of the book, as we do not include specific classes of astronomical objects as examples. Instead, we elaborate on a number of topics in fluid dynamics (some of which are applicable to astrophysics) in which a dynamical system approach may be fruitful.

My hope is that this book will encourage research in astrophysical problems, using ideas of dynamical systems and pattern theory. The second part may serve as a trigger to such new and hopefully successful approaches, as it contains descriptions of several astrophysical systems in which a dynamical system approach (using the tools introduced in the first part) has already been fruitful. If after reading this book, astrophysicists and their students are less confused about what chaos really is and in what way it is relevant (or irrelevant) to their field of research and interest, this book will have achieved its goal.

Throughout the book some references are given. These include books, review articles and original research papers. The references usually contain details, which are not explained in full in this book. In addition, they may serve as a possible source for broader and deeper study of the topics of the book. All of the references are listed alphabetically at the end of the book. This list is by no means exhaustive and I apologise to the authors of many important and relevant publications for not mentioning them.
Acknowledgements

You see how lucky you are that I’ve got so many friends?

I would like to thank all my teachers, colleagues and students (too numerous to all be mentioned here) with whom I have had the good fortune to do research on topics related to the subject of this book.

I am grateful in particular to those without whom this book would obviously have been impossible: Giora Shaviv, Robert Buchler and Ed Spiegel. They have shown me the way in scientific research and, most importantly, have taught me how to pick out the essentials from the ‘sea’ of detail. Special thanks are due to Andrew King, who has supported the idea of writing a chaos book for astrophysicists and has helped to bring it into being. Andrew, Attay Kovetz and Mike Shara read parts of the manuscript and their comments (on style and on essence) were very helpful.

Most of this book was written in France and in the United States, where I spent sabbatical leaves. I acknowledge the hospitality of the Institute d’Astrophysique de Paris, the Department of Astrophysics at the American Museum for Natural History and the Astronomy Department of Columbia University, who hosted me during these leaves. But above all it has been the warm welcome and friendship of Claude Bertout, Jean-Pierre Lasota, Mike Shara and Ed Spiegel that have made these leaves not only productive but also very pleasurable.

Over the years of writing this book my research has been supported from several sources, but most of all by continuous grants from the Israel Science Foundation; this support is gratefully acknowledged.

Finally, I would like to thank my close family for their love and support, which was an invaluable contribution towards the completion of this endeavour. My children Dani and Tamar have grown from childhood to adolescence while this book has too been (very slowly) growing. During those years I have also met Judy, who is now my wife. I dedicate this book to them.