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Excerpt
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Part 1

Fundamentals of Rotating Fluids

1

Basic Concepts and Equations for Rotating Fluids

1.1 Introduction

The special fascination of the subject of rotating fluids stems from the fact that fluid motions strongly affected by rotation are fundamentally different from those in non-rotating systems. With the motivation of explaining or predicting many atmospheric, oceanographic, planetary physical and astrophysical phenomena, the study of rotating fluids has increasingly occupied the attention of geophysicists, astrophysicists, and applied mathematicians. The subject of rotating fluids is also basic to many situations encountered by engineers and applied-fluid dynamicists in a number of important problems, ranging from centrifuges to the stability of rotating spacecraft carrying liquid payloads. Not surprisingly, a large number of theoretical, experimental, numerical, and observational studies have been made of rapidly rotating fluids over the past several decades.

Special characteristics of rotating flows lead to many inventive ideas that have been particularly and successfully applied to the theory of rotating fluids. There are primarily three special characteristics: (i) an overwhelming constraint on fluid motions imposed by controlling rotational forces, (ii) unique types of oscillatory motions, inertial oscillations and inertial waves, solely caused by the action of rotational forces, and (iii) a viscous boundary layer, produced by the effect of fast rotation, that differs markedly from that in non-rotating configurations.

These three fundamental characteristics underlie the foundation of the theory of rotating fluids, including inertial waves, rotating convection, and precessing/librating flows discussed in this monograph. Because a relatively simple mathematical solution describing inviscid wave motions can be readily obtained at leading-order proximation, theoretical progress on the corresponding viscous problems can usually be made via the elegant application of powerful asymptotic or perturbation methods.

The subject of rotating fluids contains two important but traditionally disjoint branches: inertial waves, and convective instabilities. Inertial waves describe the motion of an inviscid fluid occurring only in rotating systems, while convective motions, driven by thermal buoyancy, can take place in a viscous fluid in either rotating or non-rotating systems. Both problems, inertial waves and thermal convection, have been separately and extensively investigated. Inertial waves in rotating systems are governed by the Poincaré equation with the fluid viscosity being neglected, solutions to which in several systems have been discussed in Greenspan's monograph (Greenspan, 1968). For the problem of thermal convection, an additional equation governing the supply of buoyancy which drives convection is required. The formulation of the problem and the results of earlier research in several different geometries were presented in Chandrasekhar's monograph (Chandrasekhar, 1961). The present monograph attempts to unify the theories of inertial waves, thermal convection, and precessionally or librationally driven oscillations in the framework of an asymptotic theory that incorporates and manifests the three special characteristics of rotating fluids.

In order to illustrate the basic dynamic processes at work in rotating fluids, we expend considerable effort in studying fluid motions in rotating, closed containers completely filled with liquids without free-surface effects in three different geometries: an annular channel or a narrow-gap annulus, a circular cylinder, and a sphere or a spherical shell or an oblate spheroid. All these rotating container configurations may be either exactly or approximately realized in laboratory experiments (see, for example, Malkus, 1968; Davies-Jones and Gilman, 1971; Benton and Clark, 1974; Carrigan and Busse, 1983; Zhong et al., 1991; Kobine, 1995; Noir et al., 2001; King and Aurnou, 2013).

1.2 Equations of Motion in Rotating Systems

We first discuss briefly the full equations of motion for rotating fluids. We shall base our investigation on the continuum hypothesis throughout this book, implying that we are only concerned with length scales of the flow that are much larger than the distance between the molecules of the fluids (Batchelor, 1967). The molecular structure of the liquids is ignored and the fluids are treated as perfectly continuous and homogeneous in structure.

The continuum hypothesis allows us to define an infinitesimal element of the fluid located at the position vector $\mathbf{r} = x_i (i = 1,2,3)$, where we use cartesian indices notation, at time t. We describe the fluid element in an Eulerian system in which $\rho(\mathbf{r},t)$ describes the density of the fluid element, mass per unit volume, at position $\mathbf{r}$ and time t. The principle of conservation of mass is then expressed in indices notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_k)}{\partial x_k} = 0, \tag{1.1}$$

or in vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u}(\mathbf{r},t) = u_k(x_j,t)$ represents the velocity of the element at position $\mathbf{r}$ and time t and the total derivative D/Dt is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

Equation (1.1) or (1.2) is the partial differential equation, called the continuity equation, representing the physical law of conservation of mass under the continuum hypothesis.

Consider a Newtonian fluid of constant viscosity confined within a container that rotates non-uniformly with a time-dependent angular velocity $\mathbf{\Omega}(t)$ in the inertial frame. The principle of conservation of momentum gives rise to the equation of motion for a fluid element relative to a frame of reference. There are a number of different frames of reference that may be employed in the mathematical formulation of rotating fluids. For many geophysical problems like the dynamics of atmospheres, it is physically natural and mathematically convenient to adopt a frame of reference whose axes are fixed in a fluid-filled container – which is usually referred to as the rotating frame or the mantle frame or the body frame – so that the bounding surface of the container is stationary and only small departures from rigid-body rotation are concerned.

Denote the rate of change of any vector seen by an observer in the rotating frame by $(\partial/\partial t)_{rotating}$ and the rate of change seen by an observer in the non-rotating inertial frame by $(\partial/\partial t)_{inertial}$. The relationship between the two rates of change is

$$\left(\frac{\partial}{\partial t}\right)_{inertial} = \left(\frac{\partial}{\partial t}\right)_{rotating} + \mathbf{\Omega}(t)\times. \tag{1.3}$$

If $\mathbf{r}$ is the position vector of a fluid element, the application of Equation (1.3) gives

$$\left(\frac{\partial \mathbf{r}}{\partial t}\right)_{inertial} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{rotating} + \mathbf{\Omega}\times\mathbf{r} \ \text{ or } \ \mathbf{u}_{inertial} = \mathbf{u}_{rotating} + \mathbf{\Omega}\times\mathbf{r},$$

where $\mathbf{u}_{inertial} = (\partial\mathbf{r}/\partial t)_{inertial}$ is the velocity relative to the inertial frame and $\mathbf{u}_{rotating} = (\partial\mathbf{r}/\partial t)_{rotating}$ is the velocity measured in the rotating frame. To an observer in the inertial frame, there exists an additional term $\mathbf{\Omega}\times\mathbf{r}$ due to rotation. The acceleration in the inertial frame is then

$$\begin{aligned}\left(\frac{\partial \mathbf{u}_{inertial}}{\partial t}\right)_{inertial} &= \left[\frac{\partial\left(\mathbf{u}_{rotating} + \mathbf{\Omega}\times\mathbf{r}\right)}{\partial t}\right]_{inertial} \\ &= \left(\frac{\partial \mathbf{u}_{rotating}}{\partial t}\right)_{inertial} + \left(\frac{\partial \mathbf{\Omega}}{\partial t}\right)_{inertial}\times\mathbf{r} + \mathbf{\Omega}\times\left(\frac{\partial \mathbf{r}}{\partial t}\right)_{inertial},\end{aligned}$$

where we, after applying Equation (1.3), notice that

$$\begin{aligned}\left(\frac{\partial \mathbf{u}_{rotating}}{\partial t}\right)_{inertial} &= \left(\frac{\partial \mathbf{u}_{rotating}}{\partial t}\right)_{rotating} + \mathbf{\Omega}\times\mathbf{u}_{rotating}, \\ \left(\frac{\partial \mathbf{r}}{\partial t}\right)_{inertial} &= \mathbf{u}_{inertial} = \mathbf{u}_{rotating} + \mathbf{\Omega}\times\mathbf{r}.\end{aligned}$$

It follows then that

$$\left(\frac{\partial \mathbf{u}_{inertial}}{\partial t}\right)_{inertial} = \left[\left(\frac{\partial \mathbf{u}_{rotating}}{\partial t}\right)_{rotating} + \boldsymbol{\Omega}\times\mathbf{u}_{rotating}\right] + \left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{inertial} \times \mathbf{r} + \left[\boldsymbol{\Omega}\times\left(\mathbf{u}_{rotating} + \boldsymbol{\Omega}\times\mathbf{r}\right)\right]$$
$$= \left(\frac{\partial \mathbf{u}_{rotating}}{\partial t}\right)_{rotating} + 2\boldsymbol{\Omega}\times\mathbf{u}_{rotating} + \left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{inertial}\times\mathbf{r} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}).$$

We also notice that

$$\left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{inertial} = \left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)_{rotating}.$$

Because we shall always use the rotating frame throughout this monograph, the subscript *rotating* will be dropped unless otherwise specified. In the rotating frame of reference whose axes are fixed in a fluid-filled container, the Navier–Stokes momentum equation is of the form

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})\right] = -\nabla p + \rho\mathbf{g} + \mu\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right] + \rho\mathbf{r}\times\left(\frac{\partial\boldsymbol{\Omega}}{\partial t}\right) + \rho\mathbf{f}, \qquad (1.4)$$

where μ is the coefficient of dynamic viscosity assumed to be constant over space and time, $\mathbf{g}$ is the acceleration due to gravity, p is the pressure, the force per unit area imposed on the element of the fluid from surrounding elements, $\mathbf{u}$ is the fluid velocity relative to the rotating frame, $(\partial\boldsymbol{\Omega}/\partial t)$ represents the rate of change of the angular velocity $\boldsymbol{\Omega}(t)$, and $\mathbf{f}$ denotes an external body force. The term $(\partial\boldsymbol{\Omega}/\partial t)$, will later be explicitly worked out for different applications.

Three terms in Equation (1.4) involve angular velocity $\boldsymbol{\Omega}$. The first, $2\boldsymbol{\Omega}\times\mathbf{u}$, is called the Coriolis force; $\mathbf{r}\times(\partial\boldsymbol{\Omega}/\partial t)$ is usually referred to as the Poincaré force; and $\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$ represents the centrifugal force, which can be written in the form of a gradient

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\frac{1}{2}\nabla|\boldsymbol{\Omega}\times\mathbf{r}|^2.$$

The equation of motion (1.4) says that the rate of change of the fluid velocity in the rotating frame of reference is caused by the net joint effects of the Coriolis force, the Poincaré force, the centrifugal force, the inertial force, the pressure force, the body force, and the viscous force.

1.3 The Heat Equation

The mathematical system described by the continuity equation (1.2) and the equation of motion (1.4) is not closed because there are five scalar unknowns, ρ, p and $u_j, j = 1,2,3$ with only four equations in the system. An equation of state is needed, defining the relationship between the pressure p, the density ρ and the temperature T,

$$\rho = \rho(p, T), \tag{1.5}$$

which also introduces the local temperature T as an additional unknown. It follows that an extra equation for the conservation of energy, often called the heat equation, is required to close the mathematical system,

$$c_p \frac{DT}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \frac{Q_h}{\rho} + \frac{\mu}{2\rho}\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 - \frac{4}{3}\left(\frac{\partial u_l}{\partial x_l}\right)^2\right], \tag{1.6}$$

where k is the thermal conductivity of the liquid, Fourier's law of heat conduction

$$\mathbf{q} = -k\nabla T \tag{1.7}$$

is adopted, the material parameter μ is the coefficient of dynamic viscosity of the fluid, c_p is the specific heat at constant pressure, and Q_h denotes the rate of internal heat production per unit volume. In the heat equation the terms proportional to p and Dp/Dt, which are usually small in comparison to other terms in the equation, have been neglected. Equation (1.6) states that the rate of change of internal energy per unit mass of an element of fluid (the left-hand side) is caused by heat conduction, internal heat generation, and viscous dissipation.

1.4 The Boussinesq Equations

The three equations describing fluid motions in rotating systems, the continuity equation (1.1), the equation of motion (1.4), and the heat equation (1.6), must be simplified to be manageable in mathematical analysis. Furthermore, the equation of state (1.5) is usually highly complex and needs to be specified and simplified. In particular, these equations include very short timescale processes such as acoustic waves, a complication we do not wish to deal with and intend to remove. In typical laboratory experiments, the range of variation of temperature and pressure is small and we may treat the density as independent of the pressure and linearly dependent on the difference between the temperature, T, and a reference temperature T_0:

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right], \tag{1.8}$$

where ρ_0 is the density at T_0 and α is the coefficient of thermal expansion, which is assumed to be constant and is usually very small for many liquids, such that

$$\frac{|\rho - \rho_0|}{\rho_0} = \alpha|T - T_0| \ll 1$$

for a temperature variation of moderate amount. For many liquids, the simple equation of state (1.8) gives rise to a satisfactory approximation over a wide range of temperature and contains all the important physics with a minimum of mathematical complexity.

A commonly used approximation in both non-rotating and rotating flows, first applied by Rayleigh (1916), is the Oberbeck–Boussinesq approximation (Oberbeck, 1888; Boussinesq, 1903). The key assumption is that the variation in density is small and its effect is neglected everywhere except in the buoyancy term that drives the fluid motions. Within the Boussinesq approximation, all the thermodynamic variables such as the thermal conductivity k and the specific heat c_p can be treated as constants except for the density when multiplied by the gravity $\mathbf{g}$. At leading-order approximation, the differential in density in the continuity equation (1.2), which is of $O(\alpha)$, is neglected to yield the divergence-free condition

$$\nabla \cdot \mathbf{u} = 0, \tag{1.9}$$

which can be formally justified by scaling analysis (Spiegel and Veronis, 1960). In the framework of the Boussinesq approximation and in a reference frame fixed in a rotating container, the momentum equation (1.4), after using Equations (1.8) and (1.9), becomes

$$\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u}\right] = -\frac{1}{\rho_0}\nabla P - \mathbf{g}\alpha\Theta + \nu\nabla^2\mathbf{u} + \mathbf{r} \times \left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right) + \mathbf{f}, \tag{1.10}$$

where $\nu = \mu/\rho_0$ is called the coefficient of kinematic viscosity,

$$\Theta = T - T_0$$

denotes departure from the time-independent reference temperature T_0, and P forms the reduced pressure

$$P = p - p_0 - \frac{\rho_0}{2}\left(\boldsymbol{\Omega} \times \mathbf{r}\right) \cdot \left(\boldsymbol{\Omega} \times \mathbf{r}\right),$$

where the centrifugal force, $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$, and the hydrostatic pressure p_0 have been absorbed into the reduced pressure P. It is worth noting the following important feature relating to Equation (1.10). The solution $\mathbf{u} = \mathbf{0}$ is possible when $(\partial\boldsymbol{\Omega}/\partial t) = \mathbf{0}, \mathbf{f} = \mathbf{0}$ and $\Theta = 0$, providing a basic state for the stability analysis of thermal convection. In this case, the position of the rotation axis is not mathematically significant because the centrifugal acceleration, which depends on the location of the rotation axis, acts only to modify the pressure gradient and because the pressure does not require a boundary condition on the wall of a fluid-filled container. Here we have assumed that $\mathbf{g}$ is much larger than the centrifugal acceleration such that a simple static equilibrium without having relative fluid motion in the rotating frame is permitted.

In many geophysical and astrophysical systems, much larger ranges of temperature and pressure prevail. The convective velocities are usually small compared to the speed of sound, but are fast compared with the diffusion of heat. Under these conditions, the Boussinesq approximation remains largely valid provided the variables P, T, and $\mathbf{u}$ are regarded as departures from a well-mixed isentropic state of rest where the pressure, p_0, is hydrostatic as defined by

$$\nabla p_0 = \rho_0 \mathbf{g},$$

and the temperature T_0 follows the adiabatic gradient.

The heat equation must also be simplified. Within the Boussinesq approximation, the rate of heat production due to viscous dissipation is negligibly small in comparison to other terms in Equation (1.6), leading to the simplified heat equation

$$\frac{\partial \Theta}{\partial t} + \mathbf{u} \cdot \nabla (\Theta + T_0) = \kappa \nabla^2 (\Theta + T_0) + \frac{Q_h}{c_p \rho_0}, \tag{1.11}$$

where κ is the thermal diffusivity defined as

$$\kappa = \frac{k}{c_p \rho_0}.$$

The simplified heat equation gives an excellent approximation to Equation (1.6) for many fluids over a wide range of physical problems. These five scalar equations – the continuity equation (1.9), three scalar equations arising from the momentum equation (1.10), and the energy equation (1.11) – govern five unknown variables: the three components of velocity, $u_j, j = 1, 2, 3$, the reduced pressure P, and the temperature perturbation Θ. This set of five equations, along with a set of appropriate velocity and temperature boundary conditions, represents a mathematically closed system that describes the motion of a Boussinesq fluid in rotating systems. They are usually referred to as the Boussinesq equations, and will be studied throughout this monograph in various geometries in rotating systems.

On the bounding surface, $\mathcal{S}$, of a rotating fluid container, we must specify a set of boundary conditions for the flow velocity $\mathbf{u}$ and the temperature T. For the velocity, two types of condition are widely adopted. In a reference frame fixed in the rotating container, the first type, which is suitable for experimental studies of rotating fluids, is the no-slip boundary condition defined by

$$\hat{\mathbf{n}} \cdot \mathbf{u} = 0, \;\; \hat{\mathbf{n}} \times \mathbf{u} = \mathbf{0} \;\text{ on }\; \mathcal{S}, \tag{1.12}$$

where $\hat{\mathbf{n}} = \hat{n}_j$ denotes the normal to the bounding surface of the container $\mathcal{S}$. The second is referred to as the stress-free condition

$$\hat{n}_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = 0 \;\text{ on }\; \mathcal{S}, \tag{1.13}$$

which is appropriate for many geophysical and astrophysical systems, such as the atmosphere of a planet. There are also two types of temperature boundary condition that are widely employed in the context of thermal convection. The first is that of constant temperature, the isothermal boundary condition, on the bounding surface of the container,

$$\Theta = 0 \text{ on } \mathcal{S}. \tag{1.14}$$

The second type is that of constant heat-flux at the bounding surface

$$\hat{\mathbf{n}} \cdot \nabla\Theta = 0 \text{ on } \mathcal{S}. \tag{1.15}$$

Generally speaking, the stress-free condition (1.13) leads to a weak viscous boundary layer in rotating fluids, with simplified mathematical analysis, while no-slip condition (1.12) is usually marked by a strong viscous boundary layer along with more complex analysis.

It is worth mentioning that the Boussinesq approximation can be extended to include a basic density profile ρ_0 that is a function of space. In this case, the continuity equation (1.1) becomes

$$\nabla \cdot (\rho_0 \mathbf{u}) = 0, \tag{1.16}$$

which is usually referred to as the anelastic approximation (Batchelor, 1953; Ogura and Phillips, 1962; Gough, 1969).

1.5 The Kinetic Energy Equation

To provide physical insight into the various terms in the momentum equation (1.10), we derive the kinetic energy equation of a rotating Boussinesq fluid by taking the scalar product of Equation (1.10) with the fluid velocity $\mathbf{u}$ to get

$$\begin{aligned}\frac{1}{2}\frac{\partial|\mathbf{u}|^2}{\partial t} &= -\nabla \cdot \left[\frac{1}{2}\mathbf{u}|\mathbf{u}|^2 + \frac{P}{\rho_0}\mathbf{u} - 2\nu\mathbf{u} \times (\nabla \times \mathbf{u})\right] - 2\mathbf{u} \cdot (\boldsymbol{\Omega} \times \mathbf{u}) \\ &\quad - \nu|\nabla \times \mathbf{u}|^2 + \mathbf{u} \cdot \left[\mathbf{r} \times \left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right)\right] - \alpha \mathbf{u} \cdot \mathbf{g}\Theta, \end{aligned} \tag{1.17}$$

where we have taken the external force $\mathbf{f} = \mathbf{0}$ and made use of the divergence-free condition (1.9). Note that the Coriolis force is perpendicular to the velocity, $\mathbf{u} \cdot (\boldsymbol{\Omega} \times \mathbf{u}) = 0$, so it cannot do work. However, the Poincaré force, $\mathbf{r} \times (\partial\boldsymbol{\Omega}/\partial t)$, is fundamentally different and can do work on the fluid system.

For a Boussinesq fluid in a rotating container of volume $\mathcal{V}$ bounded by $\mathcal{S}$ with the no-slip boundary condition, the total kinetic energy of the fluid, E_{kin}, is governed by

$$\begin{aligned}\frac{\mathrm{d}E_{\text{kin}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{V}} \frac{1}{2}|\mathbf{u}|^2 \,\mathrm{d}\mathcal{V}\right) \\ &= -\int_{\mathcal{V}} \left\{\nu|\nabla \times \mathbf{u}|^2 + \mathbf{u} \cdot \left[\left(\frac{\partial \boldsymbol{\Omega}}{\partial t}\right) \times \mathbf{r}\right] + \alpha \mathbf{u} \cdot \mathbf{g}\Theta\right\} \,\mathrm{d}\mathcal{V}, \end{aligned} \tag{1.18}$$

where the no-slip boundary condition (1.12) on the bounding surface has been used. The left-hand side represents the rate of change of the kinetic energy E_{kin} of the flow. On the right-hand side of Equation (1.18), the first term, which is always negative, denotes the viscous dissipation of kinetic energy, the second term describes the kinetic energy produced by the effect of non-uniform rotation or precessing forcing, and the last term gives the rate at which buoyancy forces convert the potential energy of gravity, acting on a non-uniform density, into kinetic energy. For a viscous ($\nu \neq 0$), homogeneous ($\Theta \equiv 0$) fluid confined in a uniformly rotating container ($(\partial \mathbf{\Omega}/\partial t) = \mathbf{0}$), the total kinetic energy always decreases, $\mathrm{d}E_{kin}/\,\mathrm{d}t < 0$, from that of any initial flow because of the effect of viscous dissipation. This complex general problem in a rapidly rotating system can then be accordingly classified into three simpler cases.

1. An ideal inviscid ($\nu = 0$), homogeneous ($\Theta \equiv 0$) fluid confined in a uniformly rotating container ($(\partial \mathbf{\Omega}/\partial t) = \mathbf{0}$). In this case, the total kinetic energy is conserved, i.e., $\mathrm{d}E_{kin}/\,\mathrm{d}t = 0$. It is the Coriolis force alone that provides the restoring force for oscillatory fluid motions. The problem of inertial waves or oscillations for an inviscid fluid will be discussed in Part 2 of this monograph.
2. A viscous ($\nu \neq 0$), homogeneous ($\Theta \equiv 0$) fluid confined in a non-uniformly rotating container ($(\partial \mathbf{\Omega}/\partial t) \neq \mathbf{0}$) where the fluid motions are driven by precession or libration. This precession/libration problem for a viscous fluid will be discussed in Part 3 of this monograph.
3. A viscous ($\nu \neq 0$), unstably stratified ($\Theta \neq 0$) fluid confined in a uniformly rotating container ($(\partial \mathbf{\Omega}/\partial t) = \mathbf{0}$) where the fluid motions are driven by the buoyancy force through convective instabilities. The convection problem in rapidly rotating systems will be discussed in Part 4 of this monograph.

It will be seen that three seemingly different problems in rotating systems are mathematically and physically interconnected, and those seemingly complicated problems become mathematically simple in the framework of inertial waves or inertial oscillations that have analytical solutions in closed form. We shall discuss these three problems separately in various geometries ranging from channel through cylinder to sphere and spheroid.

1.6 Taylor–Proudman Theorem and Thermal Wind Equation

A profoundly important result in rapidly rotating systems is the Taylor–Proudman theorem (Proudman, 1916; Taylor, 1921), which can be derived from the consideration of fluid motions with characteristic velocity U in a system rotating uniformly ($(\partial \mathbf{\Omega}/\partial t) = \mathbf{0}$) with constant angular velocity $\mathbf{\Omega}$. Suppose that (i) the fluid motions are steady ($\partial \mathbf{u}/\partial t = \mathbf{0}$); (ii) the fluid motions are sufficiently slow such that the nonlinear term ($\mathbf{u} \cdot \nabla \mathbf{u}$) is much smaller than the Coriolis acceleration, or more explicitly,

$$\left| \frac{\mathbf{u} \cdot \nabla \mathbf{u}}{\mathbf{u} \times \mathbf{\Omega}} \right| = \mathrm{O}\left(\frac{U}{d\Omega} \right) = \mathrm{O}\,(Ro) \ll 1,$$