COMPUTATIONAL ORIENTED MATROIDS

Oriented matroids play the role of matrices in discrete geometry, when metrical properties, such as angles or distances, are neither required nor available. Thus they are of great use in such areas as graph theory, combinatorial optimization, and convex geometry. The variety of applications corresponds to the variety of ways they can be defined. Each of these definitions corresponds to a differing data structure for an oriented matroid, and handling them requires computational support, best realized through a functional language. Haskell is used here, and, for the benefit of readers, the book includes a primer on it. The combination of concrete applications and computation, the profusion of illustrations, many in colour, and the large number of examples and exercises will make this an ideal text for introductory courses on the subject. It will also be valuable for self-study for mathematicians and computer scientists working in discrete and computational geometry.

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Computational Oriented Matroids

Equivalence Classes of Matrices within a Natural Framework

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To Barbara, Maria, Martha and Alma

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Preface

The theory of oriented matroids is like a flourishing flower bed within the garden of discrete mathematics. Most of its plants are entwined with the many branches of other fields. It is difficult, if not impossible, to determine the multitude of independent roots. However, by examining the blossoms yielded, we can build a picture of this flower bed; reaping the fruits of the theory of oriented matroids and all its applications without requiring the spectator to be a "gardener." Thus, we hope that the reader can enjoy this developing picture and appreciate the beauty of its growth.

This book provides an introduction to oriented matroids for mathematicians, computer scientists, and engineers. It contains basic material for a course on polytopes, discrete geometry, linear programming, robotics, or any subject in which oriented matroids play a role. Software on the subject supports not only the process of learning for the student but makes the book very valuable for any specialist in the field.

Let us consider an example. Take the end points $1, 2$ of a line segment and the vertices $3, 4, 5$ of a triangle. Does the line segment intersect the triangle? This question is fundamental and decisive in computer graphics, robotics, and many other geometric problems. To answer it, many people calculate the point of intersection between the line given by the line segment and the plane given by the triangle. They decide later whether the point of intersection lies within the triangle and within the line segment. In practical applications it is typical to have many decisions of this type. Based on the theory of oriented matroids, we see the answer depends on just 5 signs, the orientations of the tetrahedra formed by ordered subsets of 4 points.

Take a (5×4)-matrix M defining in homogeneous coordinates the end points $1, 2$ of a line segment and the vertices $3, 4, 5$ of a triangle in Euclidean 3-space. The intersection property is invariant under rigid motions, and it can be expressed by "forgetting" the concept of a matrix which does change under rigid motions:

we use the *oriented matroid* of M. The intersection property depends on just five signs, the signs of the determinants of all (4×4)-submatrices of M. Up to reversing all signs simultaneously, there is precisely one solution which corresponds to the intersection case, showing that this property is invariant under rigid motions.

For mathematicians, computer scientists, and engineers matrices have become so useful in all their applications that they are reluctant to give up such a concept in order to generalize it to that of an oriented matroid. But there are many reasons to do so, especially when the matrices are used to describe *geometric objects* (such as polyhedra, convex polytopes, hyperplane arrangements, finite point sets, etc.), and the investigation concerns *combinatorial properties* of their geometric objects.

An *oriented matroid* is a generalized rigid motion invariant matrix function of high geometrical relevance like the five signs in the above example. More than three decades of exciting development in mathematics together with early contributions since 1926 have created a *theory of oriented matroids* the fundamentals of which deserve to be known by mathematicians, computer scientists, and engineers because of their applications and simplifications.

Oriented matroids even turn out to be invariant under topological transformations of the projective space that require a definition leaving the framework of Euclidean geometry. But, instead of leading to a drawback, this fact has a decisive advantage: the invariant of the projective space can always be described as a finite sequence of signs, and this in turn allows computations in terms of simple sign considerations.

For engineers working with geometric objects, for example in robotics, we recommend that their algorithms are modified to ones, where pure sign calculations can replace and simplify their computations with real numbers. Just as complex numbers have proven useful in studying real zeros of polynomials, oriented matroids play a key role in the context of geometrical problems with combinatorial properties. The generalization of those rigid motion invariant matrix functions allows an inductive generation of a complete (super-) set of geometric objects with given combinatorial properties. For example, this fact has led to the solution of a longstanding open problem in geometry concerning triangulated closed orientable two-dimensional manifolds and their flat embeddings in 3-space.

By providing independent motivations for the study of oriented matroids, throughout the mathematical history of geometry, optimization, pseudoline arrangements, molecule classification, theory of ordinary matroids, zonotopal tilings, etc., the book develops fundamentals in the theory of oriented matroids, and it presents results in the theory of oriented matroids which have not been covered in books thus far.

From the very beginning the reader is invited "to learn by playing" with various programs for oriented matroids and small-scope examples on his or her PC or on any UNIX computer in order to avoid the burden of changing the different models for oriented matroids by hand. There are many models of oriented matroids corresponding to rather different data structures. A substantial part of explanations concerns the transition between these data sets. We have used the de facto standard functional programming language Haskell 98 to explain those transitions as well as several other aspects. These functions can be evaluated. We have written a short Haskell primer that contains the restricted part of Haskell 98 that we need in this book. It is useful to have a look at this Haskell primer first when the reader hears about this functional language here for the first time.

In Chapters 1 and 2 we look at geometric matrix models in many different ways. This stance is very useful for a profound understanding of the theory of oriented matroids. In Chapter 3 we extend the combinatorial way of looking at matrices to that of oriented matroids in rank 3. We reach the arbitrary rank case in Chapter 4. Thus the reader has some motivation from previous chapters when we provide rather abstract axiom systems for oriented matroids. In Chapter 5 we study face lattices of oriented matroids, especially the convex hull concept within the theory of oriented matroids. A reverse concept forms the topic in Chapters 6 and 7. Here we start with say a face lattice that might be that of a polytope. We discuss how to find an oriented matroid the convex hull of which has the given face lattice. Chapter 8 is devoted to a central question in the theory of oriented matroids. Given an oriented matroid, can we find a matrix the oriented matroid of which is the given one? These theoretical contributions have a lot of applications in problem classes of computational synthetic geometry. We look at them in Chapter 9. Some additional applications of the theory of oriented matroids form the content of Chapter 10 and some intrinsic problems of the theory will be discussed in Chapter 11. Functional programming forms our faithful companion for carrying the burden of handling the data structure of oriented matroids. Moreover, it enables a concise and profound understanding of both transitions between different oriented matroid models and algorithmic concepts within our applications. Thanks go to several people who have commented on the manuscript at various stages. I am grateful to my son Boris who persuaded me that functional programming is a very useful tool in this context.

It was of much help to be allowed to use computer graphics, photos, models, and artwork of Benno Artmann, Andy Goldsworthy, Norman Hähn, Erich Hartmann, Karl Heinrich Hofmann, Michel Las Vergnas, and Carlo Séquin within this volume. I am grateful to Gabriela Hein for her skilled guidance during my production of all my pottery models. Above all I am very delighted that Cambridge University Press has printed all those figures in colour in which the coloured version expresses so much more than words can do.

Jürgen G. Bokowski, September 2005