1

Geometric matrix models I

A matrix will be our paradigm for explaining the concept of an oriented matroid. In this chapter the matrix

$$M := \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

will be our companion when we ask: "what does a matrix represent?"

There is a surprising multitude of definitions for an oriented matroid. Starting with a certain collection of definitions does not at the beginning really help us to understand why we should study them. Moreover, some of these definitions are so different from others that we are faced with the problem of proving their equivalence. This is seldom an easy task. The novice is in general more confused than motivated if the axioms are given without further explanation. Oriented matroids have many different historical roots. The theory grew independently for many years from various places.

Nowadays all these results have merged into one theory of oriented matroids. Some contexts in which they came up include topics like

- line and pseudoline arrangements, see Levi, 1926;
- Grassmannians, see Gutierrez Novoa, 1965;
- oriented graphs, see Las Vergnas, 1974, 1975, 1978, 1981;
- linear programming, see Bland, 1974;
- topological sphere systems, see Folkman and Lawrence, 1978;
- semispaces and configurations, see Goodman and Pollack, 1984;
- convex polytopes, see Bokowski, 1993;
- molecule classification, see Dreiding, Dress, and Haegi, 1982;
- point configurations in chemistry, see Klin, Tratch, and Treskov, 1989;

- order functions, see Kalhoff, 1996 and Jaritz, 1996;
- Platonic Solids, see Bokowski, Roudneff, and Strempel, 1997;
- hull systems, see Knuth, 1992.

to mention but a few of them. A single abstract definition of an oriented matroid, as desirable as it might be, hides from the novice its many different aspects which, as a whole, sum up to a mighty tool in discrete geometry. Thus, we have chosen another approach which will hopefully lead to a more profound understanding and better introduction: we study matrices first. Matrices, actually certain equivalence classes of them, serve as good examples for oriented matroids.

When picking a definition to begin with, we have to start with a particular concept. We may immediately find a nice idea and intriguing applications but we thereby lose the overview. In the theory of oriented matroids there are many ways to switch from one concept or model to another. Each change offers new ideas and casts new light onto a problem allowing the reader to reformulate it. Thus, for our introduction, we recommend that the reader does without a precise definition. At this early stage, we do say that we are going to generalize equivalence classes of matrices. We will find a natural framework to work with these classes. For those aquainted with the (ordinary) matroid concept and wishing to understand the theory of *oriented* matroids, we provide a reason for the word *natural*. It expresses that we do not include all matroids. Matroids have been studied extensively for other reasons before the theory of oriented matroids became an important theory in its own right. Many matroids, that is, the non-orientable ones, are canceled out in the theory of oriented matroids, thereby gaining, for example, the convexity properties of oriented matroids.

Once again, looking in more detail at familiar matrix concepts and geometrical models is our crucial starting point for a better understanding of the fundamentals in the theory of oriented matroids.

What is an $(n \times r)$-matrix of maximal rank r with real coefficients? What does it describe, what does it stand for, what does it represent?

- Is it the vector space V generated by the column vectors of the matrix?
- Does it describe the vector space in $\mathbb{R}^n$ which is orthogonal to V?
- Is it a set of vectors given by the n rows of the matrix?
- Does it describe a zonotope given by the rows of the matrix?
- Does it describe the projection of an n-cube?
- Is it a set of normal vectors of a central hyperplane arrangement?
- Does it describe a point set of homogeneous coordinates?
- Does it describe a linear map?
- Are the rows of the matrix the vertices of a convex polytope?
- Does it represent a point on a Grassmannian?
- Does it tell us all Radon partitions of a point set?

This list of questions is not complete. Nevertheless, it indicates why a definition for an oriented matroid can appear in so many forms. Linear algebra is a basic subject in many areas of mathematics and its applications. The concept of a matrix is fundamental in these contexts. A matrix leads us to a paradigm of an oriented matroid.

The oriented matroid of a matrix can be viewed as an invariant. Like the edge graph of a convex polytope, an oriented matroid is invariant under rigid motions. We can also compare an oriented matroid with the f-vector of a convex polytope. The f-vector, $f = (f_0, f_1, \ldots, f_i, \ldots)$, with components $f_0 =$ number of vertices, $f_1 =$ number of edges and in general $f_i =$ number of i-faces, is also an invariant under rigid motions of the polytope. These concepts do not even change if the convex polytope is replaced with a homeomorphic image of it, a topological ball, having the same boundary structure.

If the matrix is thought to describe a set of outer normal vectors of hyperplanes bounding a convex polytope, it also defines an invariant of this polytope, a topological ball that can also be obtained if the polytopal cell is bounded by topological hyperplanes. This way of thinking comes very close to the concept of an oriented matroid. However, the definition of an oriented matroid can be given in many contexts corresponding to the above questions and answers.

There is resistance to generalizing the concept of a matrix. If we think of the complex numbers, which have proven useful for studying zeros of polynomials, we can understand that a similar completeness property can require a more general concept. In the case of matrices this will be a topological invariant. We arrive at an invariant within projective space that cannot be represented by a matrix in all cases.

By introducing such a general concept, we gain something important. We can generate within this generalized framework all combinatorial types of abstract point configurations, with a given number of points, for a given dimension and sometimes even with certain prescribed properties. Later we can sort out those configurations that are of interest to us. We obtain with combinatorial methods an overview of all point configurations. This can be seen in many instances as the paramount advantage. Often there are no other methods available to get a similar overview.

After these general remarks, we look at matrices and what they represent.

In the first two chapters we have listed various ways to present partial or full information of a given matrix. Our catalogue is not complete but it shows some contexts in which oriented matroids appear.

Section 1.1 about convex polytopes was chosen first because oriented matroids form an essential tool for investigations in the theory of convex polytopes. The concept of polar duality as a well-known example of an anti-isomorphism of the face lattice of a convex polytope with its polar dual polytope serves us as an example. Similar isomorphisms occur in many forms in the theory of oriented matroids. Vector configurations in Section 1.2 play an essential role in problems in computational geometry. In Section 1.3 the chirotope will be the first pure data structure of the oriented matroid of a matrix. In Section 1.4 we see the polar dual aspect of Section 1.2. It will turn out later that the polarity concept cannot be carried over to the theory of oriented matroids, in general. Nevertheless, applications of oriented matroids play a role in both cases. In particular, we obtain the sphere arrangements of a matrix as an easy transition from the central hyperplane concept. Moreover, the equivalence class of homeomorphic transformations of such sphere systems leads to a second data structure of the pure oriented matroid information of a matrix. We introduce a third data structure for oriented matroids of a matrix in Section 1.6. It combines the more geometrical flavor of a sphere system with the computational advantage of a compact data structure like the chirotope. A set of hyperline sequences has proven very useful for finding extensions of a given oriented matroid with prescribed properties. Point sets on a sphere contain the matrix information up to a normalizing factor for each row vector. If we interpret these row vectors as unit normal vectors of central hyperplanes, their intersection with an affine hyperplane gives an ordered set of oriented hyperplanes. The duality concept, different from the polar dual concept, is a key tool in oriented matroid theory. We get a first understanding in Section 2.1 together with a tool that transforms the given chirotope into its dual one. We will see in our next chapter that chirotopes and sphere systems are just two different isomorphic forms in which an oriented matroid can be represented in the general case. There must not always be a corresponding matrix leading to both concepts. The pair of dual chirotopes, the pair of dual sphere systems, and the pair of dual hyperline sequences together with the pair of dual cocircuits are main building blocks for understanding the calculus of oriented matroids.

The concept of the Grassmannian in Section 2.2 is essential to an understanding of the chirotope axioms from a higher point of view. The sphere system induces a cell decomposition in the corresponding projective space. We discuss this aspect in Section 2.3. Closely related is the covector concept in Section 2.4. It describes the big face lattice of the corresponding cell decomposition. Oriented matroids are good objects to investigate zonotopes. This is the main message of Section 2.5 and in a slightly different form in Section 2.6. Cocircuits and circuits appear once more in a different setting in Sections 2.7 and 2.8. A summary section finishes our second chapter.

1.1 A convex polytope

In this section we introduce some easy concepts of convex polytopes. We try to indicate again why we do not begin immediately with a definition of an oriented matroid.

Consider the following key observation. Whereas a polytope is often given by a matrix, a data structure that does change, for example, under rigid motions, we are often interested in a property of the polytope, that is invariant under rigid motions, for example, its edge graph. If we calculate similar invariant information from the polytope, where do we lose the actual matrix information? Does the matrix lead us to a nice invariant?

In this book we will finally answer the last question in the affirmative: the oriented matroid of the matrix will contain the essential invariant information that we are interested in. However, before we understand this from a higher point of view, let us start with some aspects concerning polytopes.

We are going to investigate the transition from a polytope to its polar dual. This exemplifies how combinatorial information can have at least two representations.

$$M := \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Later in this chapter, we consider matrices on a more general level. We study not only a pair of representations of a matrix but a multitude of them. The oriented matroid will be the representation of the matrix that serves for all versions as the actual invariant form. However, the resulting data structure depends very much on the context in which the matrix was used.

We define a convex polytope P in Euclidean space $\mathbb{R}^d$ as the convex hull of a finite point set $A = \{x_1, \ldots, x_n\}$. We assume that our polytope is full dimensional, that is, it does not lie in a hyperplane.

$$P := \text{conv}\{x_1, \ldots, x_n\} := \{x | x = \sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geq 0\}.$$

If we have a hyperplane $H = \{x | \langle v, x \rangle = a\}$ in $\mathbb{R}^d$ with unit normal vector $||v|| = 1$, such that $F := H \cap P \neq \emptyset$ and $P \subset H^- := \{x | \langle v, x \rangle \leq a\}$, that is, the hyperplane touches the polytope and the closed halfspace in the opposite direction to v contains the whole polytope, we call H a *supporting hyperplane* of P and we call F a *face* of P. The *affine hull* of F aff $F := \{x | \sum_{i=1}^{d} \lambda_i y_i, \sum_{i=1}^{d} \lambda_i = 1, y_i \in F\}$ has as dimension, dim aff F, the dimension of the corresponding linear space parallel to aff F. This dimension is also used as the dimension of the face F. We have in particular the *vertices* of P as the zero-dimensional faces, the *edges* as

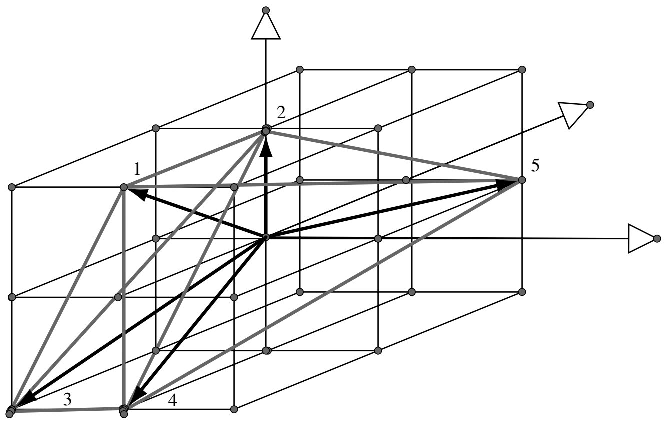

Figure 1.1 Matrix M representing a polytope

the one-dimensional faces of P, and the *facets* as the $(d-1)$-dimensional faces of P.

We reduce the set A to the *vertex set* vert $P := \{y | y$ is a vertex of $P\}$ of P. We assume that all points in A are vertices of P. We write the coordinates $x_i^1, \ldots, x_i^d$ of vertex x_i of the polytope P as the ith row of a matrix M.

$$M_{\text{aff}}(P) := \begin{array}{c} 1 \\ \\ i \\ \\ n \end{array} \begin{pmatrix} x_1^1 & x_1^2 & \ldots & x_1^d \\ \vdots & & & \vdots \\ x_i^1 & x_i^2 & \ldots & x_i^d \\ \vdots & & & \vdots \\ x_n^1 & x_n^2 & \ldots & x_n^d \end{pmatrix},$$

$$M_{\text{hom}}(P) := \begin{array}{c} 1 \\ \\ i \\ \\ n \end{array} \begin{pmatrix} 1 & x_1^1 & x_1^2 & \ldots & x_1^d \\ 1 & \vdots & & & \vdots \\ 1 & x_i^1 & x_i^2 & \ldots & x_i^d \\ 1 & \vdots & & & \vdots \\ 1 & x_n^1 & x_n^2 & \ldots & x_n^d \end{pmatrix}.$$

More often we add an additional column vector with entries 1 in the matrix interpreting the rows as homogeneous coordinates of the points. The latter matrix of a polytope will be the one that we study in detail later.

Let us now look at the polarity concept. For presenting a polytope, a more unusual way is that of providing its supporting planes.

A matrix can describe both a set of vertices of a polytope and a set of supporting planes.

If a convex polytope P in Euclidean space is given as its set of vertices in matrix form, then it is called *V-presented*, and if it is given as its set of supporting hyperplanes as a matrix, then it is called *H-presented*.

If we compare in the triangle example of Figure 1.2, the parameters of the vertices (two coordinates in each case) with the parameters of the lines (two components of the outer normal vector and the oriented distance from the origin), the numbers of parameters seem not to match, but the length of the outer normal vector of the supporting lines does not count. The better model appears if we use homogeneous coordinates, that is, if we consider convex cones generated by convex polytopes.

We use another example of a convex polytope, just a line segment S, and we consider a cone C spanned by this line segment S. We can describe the cone C by either the two vectors on the right in Figure 1.3, by the extreme rays of this cone C, or by the two rays on the left in Figure 1.3. These are extreme rays of the polar dual cone C^*, which can be interpreted as outer normal vectors of the facets of the first cone C.

It is sometimes useful to look at the matrix of a *V-presented* polytope as being the matrix of a *H-presented* polytope, or vice versa, to study properties on this *polar dual* polytope and to reinterpret results as such of the original object.

In dimension 2 this is easy to imagine and we simply learn as in the case of a regular hexagon: vertices and edges of the polar dual objects interchange.

Now we take a three-dimensional cube in a three-dimensional hyperplane H not passing through the origin in Euclidean 4-space. Assume that the center of the cube is the nearest point in H from the origin. Let H^* be the other parallel hyperplane to H with the same distance to the origin. The cube generates a four-dimensional convex cone C with apex at the origin. If we use the polar dual cone C^* of C

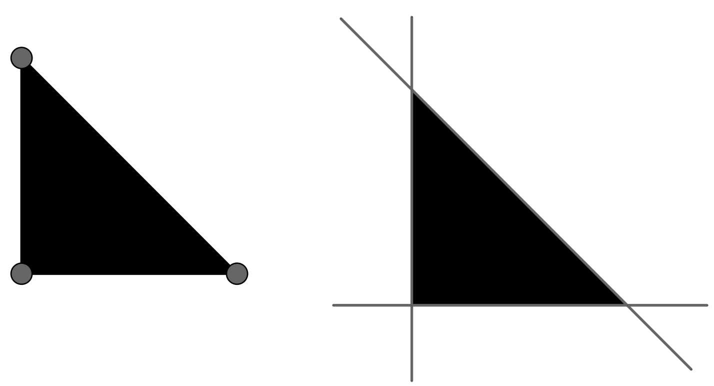

Figure 1.2 *V- and H-presented* triangle

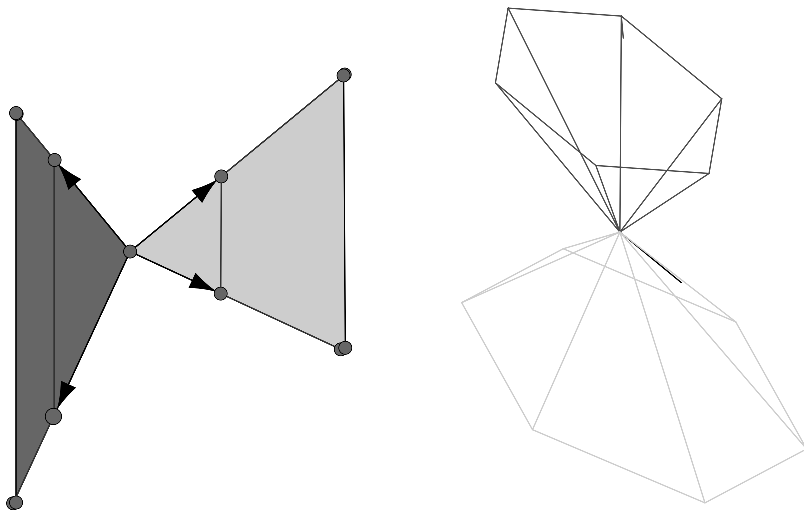

Figure 1.3 *V- and H-presented* cones, line segment, and regular hexagon

as shown in Figure 1.3 (but now in dimension 4) for which the extreme rays are formed by the outer normal vectors of facets of C, we obtain an octahedron as the intersection of the cone C^* with the hyperplane H^*. The octahedron is the polar dual convex body of the cube and vice versa, and this four-dimensional setting explains it best.

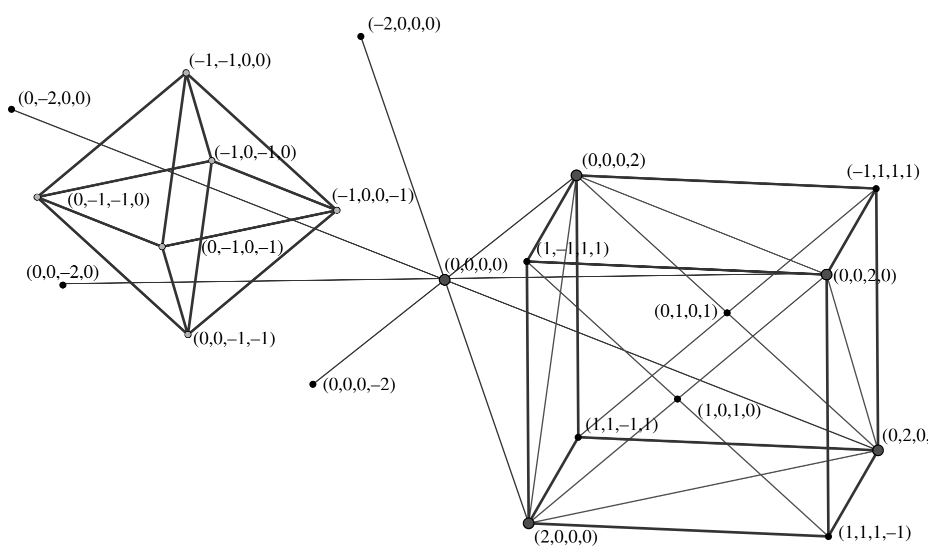

Figure 1.4 Polar duality, cube, and octahedron, projection from 4-space

The cube lies in the hyperplane $\sum x_i = 2$, the octahedron lies in the hyperplane $\sum x_i = -2$. We have used the formal definition of the polar body K^* of a non-empty, compact, convex set K with the origin in its interior:

$$K^* = \{x \in \mathbb{R}^d | \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Like the aforementioned polarity concept, there is a whole array of such transitions from various models of a matrix to others, especially if combinatorial properties of the matrix are involved. We are going to list matrix models in this chapter. Understanding these will help to reduce possible confusion for the novice when faced later with generalized concepts of matrices derived from different models.

If we study properties of convex polytopes, very often it is actually the rigid motion invariant information of that matrix, we are interested in. With the volume of the polytope, its edge graph or 1-skeleton, or more generally, with the whole face lattice of the polytope, we obtain a rigid motion invariant. The affine hulls of the facets of the polytope define a cell decomposition of the corresponding space. This cell decomposition and its face lattice (it is called the big face lattice of the polytope) is another invariant given by the polytope. If we study the cell decomposition given by the polytope, the original matrix also contains that information of the cell decomposition. It turns out that there are closely related cell decompositions that should be studied within the same framework and that cannot be represented by matrices. In that sense the matrix concept lacks a completeness property and thus has to be generalized. The idea of focussing on invariants under transformation groups, proposed by Felix Klein in his *Erlanger Programm*, guides us. We study in the following an invariant of matrices which does not even change under homeomorphic (i.e., bijective and continuous) transformations of the underlying projective space.

First, we discuss some geometric models for a matrix, or rather for equivalence classes of matrices. What are geometric or other models of a generic real $(n \times r)$-matrix? Generic here means: M has rank r and any pair of row vectors is linearly independent. We exemplify our answers for $n = 5$, $r = 3$ in the case of the following matrix M.

$$M := \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The term E is used for the ground set of elements $E = \{1, \ldots, n\}$. Although we have already mentioned that for a polytope P we prefer to study a matrix with homogeneous coordinates $M_{\text{hom}}(P)$, we can interpret the above matrix as representing a polytope with five vertices in ordinary space.

We conclude this section with a short summary. A polytope, given by a matrix, does change under rigid motions. As in the case of the edge graph of the polytope, we prefer an invariant representation. If we calculate rigid motion invariant information of the polytope, where do we lose the matrix information? A question of this type will lead us to the oriented matroid of the matrix. But for our polytope with 5 vertices the corresponding matrix has to be that where we use homogeneous coordinates:

$$M' := \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Our matrix example with 3 columns implies only lower dimensional concepts. But, of course, we have applications in mind in which even the polytope dimension exceeds the familiar number three. The interested reader can try to work in all the next sections with the above polytope matrix M'.

1.2 A vector configuration

$$M := \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We continue with perhaps the most natural *vector model* of a matrix. Interpreting the $n = 5$ rows of the matrix as vectors in $\mathbb{R}^r$, or in our example in $\mathbb{R}^3$, we obtain an ordered set of vectors v_i, $i \in E$, in $\mathbb{R}^3$. For any choice of $r = 3$ row vectors (v_j, v_k, v_l) of the matrix M, $1 \leq j < k < l \leq n$, with increasing indices, we can check whether they form a basis of $\mathbb{R}^3$, that is, whether the determinant $[j, k, l] := \det \begin{pmatrix} v_j \\ v_k \\ v_l \end{pmatrix}$ of the submatrix of M formed by the three rows j, k, l, is non-zero.

In the affirmative case, we call the ordered tuple (j, k, l) a basis and by calculating the sign of the determinant $[j, k, l]$ of the submatrix, we obtain the orientation