1

A brief introduction

Historically, this book started as a series of lectures given in the Service de Physique Théorique at Saclay, one of the lectures (Chapter 5 here) delivered by Marc Mézard. This explains the strong field theory bias adopted in its approaches and the use of some techniques rarely found in standard literature. It deals with the theory of disordered magnetic systems and for a large part of it with the Random Field Ising Model (RFIM) and the Ising Spin Glass, paradigmatic systems of frozen disorder. Such systems enjoy nontrivial properties, different from and richer than those observed in their pure (nondisordered) counterpart, that dramatically affect the thermodynamic behaviour and require specific theoretical treatment.

Disorder induces frustration and a greater difficulty for the system to find optimal configurations. Consider, for example, the case of spin glasses. These systems are dilute magnetic alloys where the interactions between spins are randomly ferromagnetic or anti-ferromagnetic. They can be modelled using an Ising-like Hamiltonian where the bonds between pairs of spins can be positive or negative at random, and with equal probability. Due to the heterogeneity of the couplings, there are many triples or loops of spin sequences which are *frustrated*, that is for which there is no way of choosing the orientations of the spins without frustrating at least one bond (Toulouse, 1977). As a consequence, even the best possible arrangement of the spins comprises a large proportion of frustrated bonds. More importantly, since there are many configurations with similar degree of frustration, one may expect the existence of many local minima of the free energy.

In mean field models the effects of frustration are enhanced, and several analytical approaches, that are extensively discussed in this book, allow an exhaustive description of the free energy landscape and the thermodynamic behaviour. For spin glasses the scenario that emerges is novel and surprising. The low temperature phase is characterized by ergodicity breaking into an ensemble of hierarchically organized pure states and the static order parameter is a function describing the structure in phase space of such an ensemble (Mézard, Parisi and Virasoro, 1987).

Interestingly, many features predicted by mean field models, such as the behaviour of susceptibilities and correlation functions or the occurrence of aging and off-equilibrium dynamics, are qualitatively observed in experiments, suggesting that the mean field scenario may hold for finite dimensional systems also. To investigate this hypothesis we analyze a field theory for the fluctuations around the mean field solution and discuss its consistency.

A transition to a glassy phase with many pure states seems to occur also in the Random Field Ising Model, where disorder is present as a random external magnetic field contrasting with the ferromagnetic spin–spin interactions. Again, an effective field theory can be developed and the effect of disorder analyzed in detail.

Before addressing in a systematic way some specific disordered models, in this brief introductory chapter we would like to discuss a few general conceptual and technical points that will often recur throughout this book.

1.1 Quenched and annealed averages

In the following we deal with spin models where the disorder is assumed to be *quenched*. What this means is that the 'disordered' variables remain fixed while the spins fluctuate. From an experimental point of view, this corresponds to a situation where the dynamical time scale of the disorder (e.g. the spin couplings in a spin glass) is much longer than the dynamical time scale of the spin fluctuations. In a given experimental sample the disordered variables assume a well defined (though unknown) time independent value. The description in terms of random variables must then be interpreted as follows: each given realization of the random variables corresponds to a given sample of the system, while the distribution according to which they are drawn describes sample to sample fluctuations. A different situation occurs when the disorder is *annealed*, that is when the time scale of the disorder and the one of spin fluctuations are comparable. In this case, in a given experimental sample, the disordered variables vary in time, their statistics being described by the corresponding distribution. The role of time scales in systems with disorder is discussed in Palmer (1982).

For quenched disorder there is a hierarchy between the spin (fast) variables and the disordered (slow) ones which is crucial in many respects. Consider, for example, the thermodynamics. Ideally one would like to compute, for a given sample, averages over the Boltzmann measure and obtain the equilibrium properties of that sample. However, due to the presence of the disorder, one can only compute quantities which are averaged also over the disorder distribution. An important question is thus to understand to what extent these averaged quantities describe the single sample physics.

It turns out that extensive observables, such as the free energy, are particularly well behaved since their associated densities are *self-averaging* in the thermodynamic limit†, that is they assume the same value for each realization of the disorder which has a finite probability. In this case sample to sample fluctuations are vanishing as the volume of the system is sent to infinity and the average value coincides with the *typical* one (i.e. the one assumed in a probable sample). On the contrary, variables that are *not* self-averaging may fluctuate widely from sample to sample and, when computing averages over disorder, rare samples with vanishing probability may give a finite contribution. The self-averageness of extensive variables means that these are the quantities one needs to compute to describe appropriately the behaviour of a single physical system. From a technical point of view, this fact makes many computations more difficult than usual. Let us consider for example the spin glass, where the disorder appears as random couplings (e.g. J_{ij} in the Ising-like model). To describe the thermodynamics of this system we may look at the free energy, which is an extensive variable and is therefore self-averaging. The free energy density f_J for a given disorder realization J is defined as

$$f_J = -\frac{1}{\beta N} \ln Z_J = -\frac{1}{\beta N} \ln \operatorname*{tr}_{\{S_i\}} \exp\{-\beta \mathcal{H}_J\{S_i\}\}, \tag{1.1}$$

where Z_J is the partition function of the model. The average value over the disorder distribution is then given by

$$f = \int \mathrm{d}J\, P(J) f_J = \overline{f_J} = -\frac{1}{\beta N} \overline{\ln Z_J}, \tag{1.2}$$

where we have indicated with an overbar the average over the disorder distribution $P(J)$. In this expression one needs to perform the average of a logarithm, which is not simple to do and quite unusual in statistical mechanics. This is a consequence of the quenched nature of the disorder, which requires us to average extensive observables like the free energy rather than, for example, the partition function itself. For this reason, Eq. (1.2) is usually referred to as a *quenched* average. Note that two distinct averages appear in Eq. (1.2) and in a precise sequence: first the thermodynamic average over the Boltzmann measure which is used to compute f_J, and then the average over the disorder. A much simpler computation is obtained by averaging directly the partition function over the disorder and then taking the logarithm

$$f_{\mathrm{an}} = -\frac{1}{\beta N} \ln \overline{Z_J} = -\frac{1}{\beta N} \ln \int \mathrm{d}J\, P(J) \operatorname*{tr}_{\{S_i\}} \exp\{-\beta \mathcal{H}_J\{S_i\}\}. \tag{1.3}$$

† This can be seen with standard thermodynamic arguments for short range models, and has also been recently proved for long range ones (Guerra and Toninelli, 2002a,b).

In this case the Boltzmann measure and the disorder distribution appear on the same footing and the two corresponding averages are performed at the same time. This procedure would be appropriate were the disorder of an annealed kind, for this reason Eq. (1.3) is referred to as an *annealed* average.

1.2 The replica method

An indirect way to deal with the logarithm appearing in the quenched average Eq. (1.2) relies on the so-called *replica method* (Kac, 1968; Edwards, 1972). This method is based on the following elementary relationship:

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n}. \tag{1.4}$$

Thanks to Eq. (1.4) the average of the logarithm is reduced to the average of Z^n. For integer n this can be expressed as the product of the partition functions of n identical copies, or *replicas*, of the original system. In this way, we have

$$\overline{\ln Z} = \lim_{n\to 0} \frac{\ln \overline{Z^n}}{n} = \lim_{n\to 0} \frac{1}{n} \ln \overline{\mathop{\mathrm{tr}}_{\{S_i^a\}} \exp\left\{-\beta \sum_a \mathcal{H}_J{}^a\{S_i^a\}\right\}}, \tag{1.5}$$

where a is a replica index. The average over the disorder appearing in the r.h.s. of (1.5) is now of a standard kind and can be carried out with simple algebra. Likewise, if one wishes to average over disorder an observable like a correlation function, e.g.

$$C_{jk} = \frac{1}{Z_J} \mathop{\mathrm{tr}}_{\{S_i\}} \exp\{-\beta \mathcal{H}_J\{S_i\}\}\, S_j S_k, \tag{1.6}$$

one needs to resort to replicas again to get rid of the J dependence of the norm. Multiplying top and bottom by Z_J^{n-1} gives

$$\begin{aligned} C_{jk} &= \frac{Z_J^{n-1}}{Z_J^n} \mathop{\mathrm{tr}}_{\{S_i\}} \exp\{-\beta \mathcal{H}_J\{S_i\}\}\, S_j S_k \\ &= Z_J^{-n} \mathop{\mathrm{tr}}_{\{S_i^a\}} \exp\left\{-\beta \sum_a \mathcal{H}_J{}^a\{S_i^a\}\right\} S_j^1 S_k^1, \end{aligned} \tag{1.7}$$

and finally

$$\overline{C_{jk}} = \lim_{n\to 0} \overline{\mathop{\mathrm{tr}}_{\{S_i^a\}} \exp\left\{-\beta \sum_a \mathcal{H}_J{}^a\{S_i^a\}\right\} S_j^1 S_k^1}. \tag{1.8}$$

Under disorder averaging, disorder with independent replicas is replaced by *coupled* replicas. The task is then to compute properties of the system with the effective fields and couplings resulting from J-averaging (ϕ_i^{ab} for spin glasses, related to the spin overlap $S_i^a S_i^b$ and ϕ_i^a for the Random Field Ising Model, related to S_i^a) and analytically continue the result to $n = 0$.

In spin glasses, as we shall see, even the mean field approximation is highly nontrivial, due to the matrix nature of the order parameter Q_i^{ab}, the thermal average of the field ϕ_i^{ab}. It turns out that the mean field solution may break the invariance with respect to replica permutations, endowed by the original replicated Hamiltonian. In this case we say that Replica Symmetry Breaking (RSB) occurs. For RSB solutions, deciding which is the correct pattern of symmetry breaking, i.e. what is the structure of the overlap matrix Q^{ab} in the replica space, is a demanding task. We will discuss in detail the correct ansatz for Q^{ab} and the novel physical scenario it describes. This complex RSB structure makes the analysis of the Gaussian fluctuations around the mean field solution much more complicated than in standard systems, and new techniques must be introduced to deal with the inversion of the Hessian matrix.

In the Random Field Ising Model, on the other hand, the mean field solution is trivial since the order parameter bears only one replica index. In this case the treatment of the model in finite dimension is simpler and a perturbative renormalization group can easily be performed. The glassy phase is, however, more difficult to detect, requiring a more sophisticated analysis of the dependence of the free energy on two-point functions.

1.3 The generating functional

So far, we have discussed the conceptual and technical problems originated by the presence of the quenched disorder in static computations. The same kind of difficulties arise when adopting a dynamical approach. Let us consider, for example, a Langevin kind of dynamics, which is the one mostly used in analytical computations. In this case the dynamical evolution of a given field $\phi_i(t)$ is determined by the following stochastic equation:

$$\mathcal{E}_i^J\{\phi_i(t)\} \equiv \frac{\partial \phi_i\,(t)}{\partial t} + \frac{\partial \mathcal{H}_J\,\{\phi_i\}}{\partial \phi_i} - \eta_i\,(t) = 0, \tag{1.9}$$

where $\eta_i(t)$ represents a Gaussian thermal noise. Here, again, two noises appear (the thermal noise and the quenched disorder) and two averages must be performed. In principle, one should first compute, for a given disorder instance, the dynamical observables by integrating out the thermal noise. Then, the result must be averaged over the distribution of the disorder. The *generating functional* approach (Martin, Siggia and Rose, 1973) is a technique which allows us to do it all, in a way that bears resemblance to the static computations. The main idea is to introduce a dynamical functional using the identity

$$1 = \hat{Z} = \int \prod_i \mathrm{D}\phi_i(t)\, \delta\left(\mathcal{E}_i^J\{\phi_i(t)\}\right)\, \left|\det \partial_j \mathcal{E}_i^J\right|, \tag{1.10}$$

where $\mathrm{D}\phi_i(t)$ stands for functional integration over the field $\phi_i(t)$.

In the presence of appropriate external sources $\hat{Z}$ becomes then the generating functional of dynamical averages (see Chapter 3). Both the delta function and the determinant are expressed using integral representations and the generating functional is written in terms of a dynamical (disorder dependent) Lagrangian which plays for the dynamics a role analogous to the replicated Hamiltonian in the statics (summations over replicas being replaced by integrals over time). At this point, because the norm $\hat{Z}$ is J independent (in contrast to Z_J for the static case) one can trivially perform the average over quenched disorder and over thermal noise. The result is the effective generating functional for correlations and responses (the analogues of the static overlap matrix). As we have seen, in the static computation of free energy, the average over disorder generates a coupling between distinct replicas. In the dynamical context there are no replicas, and the effect of the disorder is to generate nonlocality in time, i.e. a coupling between distinct *times*. In statics, the order parameter may break the replica permutation symmetry and exhibit a nontrivial structure in replica space. In dynamics, correlation and response functions may in some regimes break the time translation invariance and exhibit nonstandard patterns of dynamical evolution where the fluctuation dissipation theorem is violated. We shall discuss such a scenario in detail for a simple spin glass model.

1.4 General comments

The difficulties related to the analysis of disordered models stem mainly from the complex nature of the order parameter and the existence of a glassy phase. In the context of the replica method, this is already manifest at mean field level, where, below the transition to the glassy phase, the saddle point acquires an RSB structure. From a dynamical point of view, time translational invariance is lost and equilibrium never reached. The analysis in finite dimension becomes rather complicated, since even the computation of Gaussian fluctuations around an RSB mean field solution is not a simple problem. In this book, we deal with two classics of disordered models: spin models with disordered magnetic field (the Random Field Ising Model) and spin models with disordered exchange couplings (spin glasses). For these two cases, the above difficulties affect our analysis in different respects:

(i) For the Random Field Ising Model, we are mostly interested in understanding the nature of the transition between the ferromagnetic and the paramagnetic state and computing the critical point properties. To do that, we approach the transition from above, always remaining in a replica symmetric region. (Correspondingly, the dynamics is of an equilibrium, time translational invariant, kind.) In this phase a renormalization group analysis, both static and dynamic, up to one loop can be carried out. The presence of bound states can be investigated exactly at the ferro–para transition, when and if the theory is still replica symmetric. However, the vitreous phase is not directly addressed.

(ii) For spin glasses, a mean field analysis reveals a rich low temperature phase which can be described in detail. Our main aim is then to study the stability of the mean field scenario in finite dimension. To do that, we place ourselves in the low temperature region and develop a field theory for the fluctuations around the mean field RSB solution. Since we now deal with a replica symmetry broken theory, we mainly analyze the Gaussian fluctuations and obtain the free propagators. One-loop corrections in the glassy phase are only dealt with for the equation of state, relying upon scaling arguments to hint at behaviour away from the upper critical dimension. Renormalization group calculations are carried out at the critical temperature.

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2

The Random Field Ising Model

The Random Field Ising Model (RFIM) represents one of the simplest models of cooperative behaviour with quenched disorder, and it is, in a way, complementary to the Ising Spin Glass which will be extensively treated later in this book. It accounts for the presence of a random external magnetic field which antagonizes the ordering induced by the ferromagnetic spin–spin interactions. From an experimental point of view, on the other hand, as shown by Fishman and Aharony (1979) and Cardy (1984), it is equivalent to a dilute anti-ferromagnet in a *uniform* field (see Belanger, 1998 for a recent review on experimental results).

Despite twenty-five years of active and continuous research the RFIM is not yet completely understood. The problem seems related to the presence of bound states in the ferromagnetic phase, which make the standard theoretical approaches not adequate to analyze the critical behaviour. Here we discuss the RFIM in the context of perturbative field theory. The chapter is organized as follows: in Section 2.1 we define the model and outline the main expectations for its qualitative behaviour. In Section 2.2 we introduce an effective replicated ϕ^4 field model where the disorder has been integrated out. Then we perform a perturbative analysis on this model (Section 2.3) and illustrate how the so-called *dimensional reduction* arises (Section 2.4). Finally, in Section 2.5 we introduce some generalized couplings which need to be taken into account to properly describe the system; we perform a perturbative Renormalization Group (RG) close to the upper critical dimension (Sections 2.6 and 2.7) and discuss the occurrence of a vitrous transition (Section 2.8).

We leave aside several other approaches used to treat this model such as real space RG, high temperature expansions, Monte Carlo simulations, etc. For those we refer the reader to the review of Natterman (1998). We should also mention some recent and interesting work by Tarjus and Tissier (2003) that uses the functional RG in a very promising way.

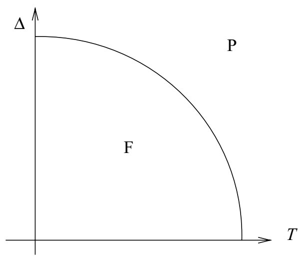

Figure 2.1 Qualitative phase diagram for the RFIM with zero external homogeneous field: P indicates the paramagnetic phase, F the ferromagnetic one

2.1 The model

The Hamiltonian of the RFIM is analogous to the one of the classical Ising Model, but allowing for a disordered quenched magnetic field:

$$-\mathcal{H} = \sum_{(ij)} J_{ij}\, S_i S_j + \sum_i h_i\, S_i. \tag{2.1}$$

Here $S_i = \pm 1,\ J_{ij} = J$ for nearest neighbour pairs (i, j) and the h_i are quenched random variables drawn with a Gaussian distribution defined by

$$\overline{h_i} = 0, \qquad \overline{h_i h_j} = \Delta\, \delta_{i;j}. \tag{2.2}$$

Note that, because of the presence of the quenched disorder, we deal from now on with two different kinds of average: the thermal average over the Boltzmann measure and the quenched average over the disorder distribution. To distinguish them we shall indicate the first with brackets $\langle \cdots \rangle$ and the second with an overbar $\overline{\cdots}$ (as in (2.2)).

In the pure case, i.e. for the Ising Model with no external magnetic field, a second order transition exists at temperature T_c^0, separating a high temperature paramagnetic phase from a low temperature ferromagnetic one. The presence of a random external magnetic field clearly disturbs the ordering effect associated with the ferromagnetic exchange interactions: thus one expects a decrease of the transition temperature with increasing disorder strength Δ. Qualitatively, then the phase diagram exhibits a paramagnetic phase for large Δ, and/or large temperature T, and a ferromagnetic phase in the opposite limits (see Fig. 2.1).

At low enough dimensions the action of the random field can inhibit the creation of the ordered phase. A quite robust argument has been given by Imry and Ma (1975). It estimates how a random field can destroy a predominantly ferromagnetic environment. Consider a domain of size R in a ferromagnetic region (see Fig. 2.2) and reverse the spins inside it. The energy cost due to the exchange interactions E_J is proportional to the surface of the domain and is therefore of order $J R^{D-1}$,

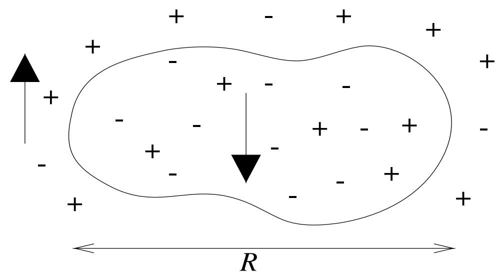

Figure 2.2 Domain of size R in a ferromagnetic environment

where D is the dimension of the physical system. The Zeeman energy associated with the random field E_{RF} is, according to the central limit theorem, $E_{\mathrm{RF}}^2 \sim \Delta R^D$. The global energy balance is then written

$$E(R) \approx J R^{D-1} - \sqrt{R^D \Delta}, \tag{2.3}$$

and the fluctuations of h_i will always destroy the ferromagnetic state if

$$\frac{D}{2} > D - 1, \qquad \text{i.e.} \qquad D < 2. \tag{2.4}$$

2.2 The replicated field theory

It is convenient to recast the Hamiltonian (2.1) into a soft spin version. This can easily be done by writing, within a constant, the partition function Z as:

$$Z = \int \prod_i (\mathrm{D}\phi_i) \operatorname*{tr}_{\{S_i\}} \left\{ \exp\left[-\frac{1}{2\beta} \sum_{(ij)} \phi_i (J^{-1})_{ij} \phi_j + \sum_i (\phi_i + \beta h_i) S_i \right] \right\}, \tag{2.5}$$

where $\mathrm{D}\phi_i$ stands for $\mathrm{d}\phi_i/\sqrt{2\pi}$. By taking the trace over the spins, we then get

$$Z = \int \prod_i (\mathrm{D}\phi_i) \exp\left\{ -\frac{1}{2\beta} \sum_{i,j} \phi_i (J^{-1})_{ij}\, \phi_j + \sum_i \ln\cosh\, (\phi_i + \beta h_i) \right\}. \tag{2.6}$$

If we assume that the argument of the cosh is small, develop it and retain the relevant terms, after appropriate rescaling, we can in turn rewrite (2.6) as a ϕ_i^4 Lagrangian in a random field h_i:

$$\begin{aligned} \mathrm{e}^{-F\{h_i\}} \equiv Z = \int \prod_i (\mathrm{D}\phi_i)\ \exp\Bigg\{ &-\frac{1}{2} \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \phi(p)[p^2 + r_0]\, \phi(-p) \\ &- u_1 \sum_i \frac{\phi_i^4}{4!} + \sum_i h_i\, \phi_i \Bigg\}, \end{aligned} \tag{2.7}$$