Part I

Path integrals for quantum mechanics in curved space

1
Introduction to path integrals

Path integrals play an important role in modern quantum field theory. One usually first encounters them as useful formal devices to derive Feynman rules. For gauge theories they yield straightforwardly the Ward identities. Namely, if BRST symmetry (the "quantum gauge invariance" discovered by Becchi, Rouet, Stora and Tyutin [14]) holds at the quantum level, certain relations between Green functions (Ward indentities) can be derived from path integrals, but details of the path integral (for example, the precise form of the measure) are not needed for this purpose.[1] Once the BRST Ward identities for gauge theories have been derived, unitarity and renormalizability can be proven, and at this point one may forget about path integrals if one is only interested in perturbative aspects of quantum field theories. One can compute higher-loop Feynman graphs without ever using path integrals.

However, for nonperturbative aspects, path integrals are essential. The first place where one encounters path integrals in nonperturbative quantum field theory is in the study of instantons and solitons. Here advanced methods based on path integrals have been developed. For example, in the case of instantons the correct measure for integration over their collective coordinates (corresponding to the zero modes) is needed. In particular, for supersymmetric nonabelian gauge theories, there are only contributions from these zero modes, while the contributions from the nonzero modes cancel between bosons and fermions. Another area where the path integral

[1]To prove that the BRST symmetry is free from anomalies, one may either use regularization-free cohomological methods, or one may perform explicit loop graph calculations using a particular regularization scheme. When there are no anomalies, but the regularization scheme does not preserve the BRST symmetry, one can always add local counterterms to the action at each loop level to restore the BRST symmetry. In these manipulations the path integral measure is usually not taken into account.

measure is important is quantum gravity. In particular, in modern studies of quantum gravity based on string theory, the measure is crucial in obtaining the correct correlation functions.

One can compute path integrals at the nonperturbative level by going to Euclidean space, discretizing the path integrals on lattices and using powerful computers. In this book we use a continuum approach. We study a class of simple models which lead to path integrals in which no infinite renormalization is needed, but some individual diagrams are divergent and need be regulated, and subtle issues of regularization and measures can be studied explicitly. These models are the quantum mechanical (one-dimensional) nonlinear sigma models. The one- and two-loop diagrams in these models are power-counting divergent, but the infinities cancel in the sum of diagrams for a given process at a given loop level.

Quantum mechanical (QM) nonlinear sigma models can be described by path integrals and are toy models for realistic path integrals in four dimensions. They describe curved target spaces and contain double-derivative interactions (quantum gravity has also double-derivative interactions). The formalism for path integrals in curved space has been discussed in great generality in several books and reviews [15–26]. In the first half of this book we define the path integrals for these models and discuss various subtleties. However, quantum mechanical nonlinear sigma models can also be used to compute anomalies of realistic four- and higher-dimensional quantum field theories, and this application is thoroughly discussed in the second half of this book. Furthermore, quantum mechanical path integrals can be used to compute correlation functions and effective actions. For references in flat space see [27], and for some work in curved space see [28–30].

The study of path integrals in curved space was studied in detail by DeWitt [15]. He first extended to curved space a result of Pauli [16] for the transition element for infinitesimal times which was the product of the exponent of the classical action evaluated for a classical trajectory, times the Van Vleck–Morette determinant [17]. He verified that this transition element satisfied a Schrödinger equation with Hamiltonian $\hat{H} + \frac{1}{12}\hbar^2 R$ ($-\frac{1}{12}\hbar^2 R$ in our conventions for R), where $\hat{H} = \frac{1}{2}\hat{g}^{-1/4}\hat{p}_i\hat{g}^{ij}\hat{g}^{1/2}\hat{p}_j\hat{g}^{-1/4}$. He also claimed that this transition element could be written as a path integral with a modified action, which was the sum of the classical action and a term $+\frac{\hbar^2}{12}R$. The latter term comes from the Van Vleck determinant.[2] His work has led to an enormous literature on this subject, with many authors proposing various ingenuous definitions or approximations of the

[2]There exists some confusion in the literature about the coefficient of R in the action in the path integral for the transition element related to the minimal hamiltonian operator $\hat{H}$ ("the counter term with R"). Initially DeWitt obtained $\frac{1}{6}$ [15]. However, recently in [26] he rectified this to $\frac{1}{8}$, a result with which we agree, at least if one uses the regularization schemes discussed in this book, see eqs. (2.81), (3.73), (4.28) and Appendix B. (Note: some of these schemes have additional noncovariant $\Gamma\Gamma$ terms.)

infinitesimal transition element, and various proposals for iterations which should produce the finite transition amplitude, see for example [31–34].

In Part I of this book we show how to define and compute the transition element for finite times using path integrals. This yields, in particular, the transition element for infinitesimal times in a series expansion. Path integrals are of course just one of many ways of computing the transition element, but for the calculation of anomalies the path integral method is far superior as we hope to demonstrate in this book.

1.1 The simplest case: a particle in flat space

Before considering path integrals in curved space, we first review the simple case of a nonrelativistic particle moving in an n-dimensional flat space and subject to a scalar potential $V(x)$. We are going to derive the path integral from the canonical (operatorial) formulation of quantum mechanics. We will also compute the transition amplitude in the free case (i.e. with vanishing potential), a useful result to compare with when we deal with the more complicated case of curved space.

Thus, let us consider a particle with coordinates x^i, conjugate momenta p_i and mass m. As the quantum Hamiltonian we take

$$H(\hat{x}, \hat{p}) = \frac{1}{2m}\hat{p}_i \hat{p}^i + V(\hat{x}) \tag{1.1}$$

where, as usual, hats denote quantum mechanical operators. We are interested in deriving a path integral representation of the transition amplitude

$$T(z, y; \beta) \equiv \langle z | e^{-\frac{\beta}{\hbar}\hat{H}} | y \rangle \tag{1.2}$$

for the particle to propagate from the point y^i to the point z^i in a Euclidean time β. We use a language appropriate to quantum mechanics ("transition amplitude", etc.) even though we consider a Euclidean approach. The usual quantum mechanics in Minkowskian time is obtained by the substitution $\beta \to it$, which corresponds to the so-called Wick rotation, an analytical continuation in the time coordinate that relates statistical mechanics to quantum mechanics, and vice versa.

We use eigenstates $|x\rangle$ and $|p\rangle$ of the position operator $\hat{x}^i$ and momentum operator $\hat{p}_i$, respectively,

$$\hat{x}^i |x\rangle = x^i |x\rangle, \quad \hat{p}_i |p\rangle = p_i |p\rangle, \tag{1.3}$$

together with the completeness relations

$$I = \int d^n x \, |x\rangle\langle x| = \int d^n p \, |p\rangle\langle p| \tag{1.4}$$

and the scalar products

$$\langle x_1 | x_2 \rangle = \delta^n(x_1 - x_2), \quad \langle p_1 | p_2 \rangle = \delta^n(p_1 - p_2), \quad \langle x | p \rangle = \frac{1}{(2\pi\hbar)^{n/2}} e^{\frac{i}{\hbar} p_i x^i}. \tag{1.5}$$

It is easy to show that the transition amplitude should satisfy the Schrödinger equation (see (2.229) and (2.230))

$$-\hbar\frac{\partial}{\partial\beta}T(z,y;\beta) = H(z)T(z,y;\beta) \tag{1.6}$$

with the boundary condition

$$T(z,y;0) = \delta^n(z-y) \tag{1.7}$$

where the Hamiltonian in the coordinate representation is, of course, given by

$$H(z) = -\frac{\hbar^2}{2m}\frac{\partial}{\partial z^i}\frac{\partial}{\partial z_i} + V(z). \tag{1.8}$$

A similar equation holds at the point y^i.

The derivation of a path integral representation for the transition amplitude is rather standard. The transition amplitude can be split into N factors

$$\begin{aligned} T(z,y;\beta) &= \langle z|\left(\mathrm{e}^{-\frac{\beta}{\hbar N}\hat{H}}\right)^N|y\rangle = \langle z|\underbrace{\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}\cdots\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}}_{N\text{ times}}|y\rangle \\ &= \int\left(\prod_{k=1}^{N-1}d^nx_k\right)\prod_{k=1}^{N}\langle x_k|\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle \end{aligned} \tag{1.9}$$

where we have denoted $x_0^i = y^i$, $x_N^i = z^i$, $\epsilon = \beta/N$, and used $N-1$ times the completeness relations with position eigenstates. Then one can use N times the completeness relations with momentum eigenstates and obtain

$$T(z,y;\beta) = \int\left(\prod_{k=1}^{N-1}d^nx_k\right)\left(\prod_{k=1}^{N}d^np_k\right)\prod_{k=1}^{N}\langle x_k|p_k\rangle\langle p_k|\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle. \tag{1.10}$$

This is still an exact formula, but we are now going to evaluate it using approximations which are correct in the limit $N\to\infty$ $(\epsilon\to 0)$. The key point for deriving the path integral is to evaluate the following matrix element

$$\begin{aligned} \langle p|\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})}|x\rangle &= \langle p|\left[1-\frac{\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})+\cdots\right]|x\rangle \\ &= \langle p|x\rangle - \frac{\epsilon}{\hbar}\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle + \cdots \\ &= \langle p|x\rangle\left[1-\frac{\epsilon}{\hbar}H(x,p)+\cdots\right] \\ &= \langle p|x\rangle\,\mathrm{e}^{-\frac{\epsilon}{\hbar}H(x,p)+\cdots}. \end{aligned} \tag{1.11}$$

The replacement $\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle = \langle p|x\rangle\,H(x,p)$ follows from the simple structure of the Hamiltonian in (1.1), which allows to act with the position and momentum operators on the corresponding eigenstates, so that these operators are simply replaced by the corresponding eigenvalues. In this way the Hamiltonian operator $\hat{H}(\hat{x},\hat{p})$ is replaced by the Hamiltonian function $H(x,p)=p^2/2m+V(x)$. These approximations are justified in the limit $N\to\infty$ for many physically interesting potentials (i.e. the "dots" in (1.11) can be neglected in this limit), in which cases a rigorous mathematical proof is also available, and goes under the name of the "Trotter formula" [21]. Finally, using the expression for $\langle x|p\rangle$ given in (1.5), and recalling that $\langle p|x\rangle = \langle x|p\rangle^*$, one obtains

$$\langle x_k|p_k\rangle\langle p_k|\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle = \frac{1}{(2\pi\hbar)^n}\mathrm{e}^{\frac{i}{\hbar}p_k\cdot(x_k-x_{k-1})-\frac{\epsilon}{\hbar}H(x_{k-1},p_k)} \tag{1.12}$$

which can now be inserted into (1.10). At this point the expression of the transition amplitude does not contain any more operators, and reads as

$$\begin{aligned} T(z,y;\beta) &= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}d^nx_k\right)\left(\prod_{k=1}^{N}\frac{d^np_k}{(2\pi\hbar)^n}\right) \\ &\quad\times\exp\left\{-\frac{\epsilon}{\hbar}\sum_{k=1}^{N}\left[-ip_k\cdot\frac{(x_k-x_{k-1})}{\epsilon}+H(x_{k-1},p_k)\right]\right\} \\ &= \int Dx\,Dp\,\mathrm{e}^{-\frac{1}{\hbar}S[x,p]}. \end{aligned} \tag{1.13}$$

This is the path integral in phase space. We recognize in the exponent a discretization of the classical Euclidean phase space action

$$\begin{aligned} S[x,p] &= \int_0^\beta dt\,[-ip\cdot\dot{x}+H(x,p)] \\ &\to \epsilon\sum_{k=1}^{N}\left[-ip_k\cdot\frac{(x_k-x_{k-1})}{\epsilon}+H(x_{k-1},p_k)\right] \end{aligned} \tag{1.14}$$

where again $\beta=N\epsilon$. The last line in (1.13) is symbolic and indicates a formal sum over paths in phase space weighted by the exponential of minus their classical action.

The configuration space path integral is easily derived by integrating out the momenta in (1.13). Completing squares and using Gaussian integration one obtains

$$\begin{aligned} T(z,y;\beta) &= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}d^nx_k\right)\left(\frac{m}{2\pi\hbar\epsilon}\right)^{nN/2} \\ &\quad\times\exp\left\{-\frac{\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\left(\frac{x_k-x_{k-1}}{\epsilon}\right)^2+V(x_{k-1})\right]\right\} \\ &= \int Dx\,\mathrm{e}^{-\frac{1}{\hbar}S[x]}. \end{aligned} \tag{1.15}$$

This is the path integral in configuration space. In the exponent one finds a discretization of the classical Euclidean configuration space action

$$S[x] = \int_0^\beta dt \left[\frac{m}{2}\dot{x}^2 + V(x)\right]$$
$$\to \epsilon \sum_{k=1}^{N} \left[\frac{m}{2}\Big(\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 + V(x_{k-1})\right]. \quad (1.16)$$

Again the last line in (1.15) is symbolic, and indicates a sum over paths in configuration space.

For the case of a vanishing potential, the path integral can be evaluated exactly [45, 46, 21]. Performing successive Gaussian integrations one obtains

$$T(z, y; \beta) = \Big(\frac{m}{2\pi\hbar\beta}\Big)^{n/2} \mathrm{e}^{-m(z-y)^2/2\beta\hbar}. \quad (1.17)$$

This final result is very suggestive. Up to a prefactor, it consists of the exponential of the classical action evaluated on the classical trajectory. This is typical for the cases where the semiclassical approximation is exact. The prefactor can be considered as containing the "one-loop" corrections which make up the full result (thus "semiclassical" = "classical + one-loop") [21].

The preceding approach is called time slicing, and will be applied to nonlinear sigma models (models in curved target space) in Chapter 2. In Chapters 3 and 4 we shall use two other equivalent methods of computing path integrals: mode regularization and dimensional regularization. In these cases we shall use a somewhat different way to evaluate path integrals. We expand the continuous paths $x^i(t)$ into a fixed classical "background" part $x^i_{bg}(t)$ plus "quantum fluctuations" $q^i(t)$

$$x^i(t) = x^i_{bg}(t) + q^i(t). \quad (1.18)$$

Here $x^i_{bg}(t)$ is a fixed function: it solves the classical equations of motion and takes into account the boundary conditions ($x^i(0) = y^i$ and $x^i(\beta) = z^i$). "For a free particle one has"

$$x^i_{bg}(t) = y^i + (z^i - y^i)\frac{t}{\beta}, \quad (1.19)$$

while the arbitrary fluctuations $q^i(t)$ vanish at the boundaries. One may interpret $x^i_{bg}(t)$ as the origin and $q^i(t)$ as the coordinates of the "space of paths".

Now one can compute the path integral (1.15) for a vanishing potential

$$T(z, y; \beta) = \int Dx\, \mathrm{e}^{-\frac{1}{\hbar}S[x]} = \int D(x_{bg} + q)\, \mathrm{e}^{-\frac{1}{\hbar}S[x_{bg}+q]}$$

$$
\begin{aligned}
&= \int Dq\, \mathrm{e}^{-\frac{1}{\hbar}(S[x_{bg}]+S[q])} = \mathrm{e}^{-\frac{1}{\hbar}S[x_{bg}]} \int Dq\, \mathrm{e}^{-\frac{1}{\hbar}S[q]} \\
&= A\mathrm{e}^{-\frac{1}{\hbar}S[x_{bg}]} = A\mathrm{e}^{-\frac{m(z-y)^2}{2\beta\hbar}} \qquad (1.20)
\end{aligned}
$$

where we have used the translational invariance of the path integral measure $Dx = D(x_{bg} + q) = Dq$ (at the discretized level this is evident from writing $d^n x_k = d^n(x_{k,bg} + q_k) = d^n q_k$) and the fact that in the action there is no term linear in q^i (the action is quadratic in q^i, but the term linear in q^i must also be linear in x_{bg}, but then this term must vanish by the equations of motion). Finally, the constant $A = \int Dq\, \exp(-\frac{1}{\hbar}S[q])$ is not determined by this method, but it can be fixed by requiring that (1.20) solves the Schrödinger equation (1.6) with the boundary condition in (1.7). The value $A = (m/2\pi\hbar\beta)^{n/2}$ is sometimes called the Feynman measure.

1.2 Quantum mechanical path integrals in curved space require regularization

The path integrals for the quantum mechanical systems we shall discuss have a Hamiltonian $\hat{H}(\hat{x}, \hat{p})$ which is more general than $\hat{T}(\hat{p}) + \hat{V}(\hat{x})$. We shall typically be considering models with a Euclidean Lagrangian of the form $L = \frac{1}{2}g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt} + iA_i(x)\frac{dx^i}{dt} + V(x)$, where $i, j = 1, \ldots, n$. These systems are one-dimensional quantum field theories with double-derivative interactions, and hence they are not ultraviolet finite by power counting; rather, the one- and two-loop diagrams are divergent as we shall discuss in detail in the next section. The ultraviolet infinities cancel in the sum of diagrams, but one needs to regularize individual diagrams which are divergent. The results of individual diagrams are then regularization-scheme dependent, and also the results for the sum of diagrams are finite but scheme dependent. One must then add finite counterterms which are also scheme dependent, and which must be chosen such that certain physical requirements are satisfied (renormalization conditions). Of course, the final physical answers should be the same, no matter which scheme one uses. Since we shall be working with actions defined on a compact time-interval, there are no infrared divergences. We shall also discuss nonlinear sigma models with fermionic point particles $\psi^a(t)$ with again $a = 1, \ldots, n$. Also one- and two-loop diagrams containing fermions can be power-counting divergent. For applications to chiral and gravitational anomalies the most important cases are the rigidly supersymmetric models, in particular the quantum mechanical models with $N = 1$ and $N = 2$ supersymmetry, but nonsupersymmetric models with or without fermions will also be used as they are needed for applications to trace anomalies.

Quantum mechanical path integrals can be used to compute anomalies of n-dimensional quantum field theories. This was first shown by Alvarez-Gaumé and Witten (AGW) [1, 35, 36], who studied various chiral and gravitational anomalies (see also [37, 38]). Subsequently, Bastianelli and van Nieuwenhuizen [39, 40] extended their approach to trace anomalies. With the formalism developed below one can now, in principle, compute any anomaly, and not only chiral anomalies. In the work of Alvarez-Gaumé and Witten, the chiral anomalies themselves were written directly as a path integral in which the fermions have periodic boundary conditions. Similarly, the trace anomalies lead to path integrals with antiperiodic boundary conditions for the fermions. These are, however, only special cases, and in our approach any Jacobian will lead to a corresponding set of boundary conditions.

Because chiral anomalies have a topological character, one would expect details of the path integral to be unimportant and only one-loop graphs on the worldline to contribute. In fact, in the approach of AGW this is indeed the case.[3] On the other hand, for trace anomalies, which have no topological interpretation, the details of the path integral do matter and higher loops on the worldline contribute. In fact, it was precisely because three-loop calculations of the trace anomaly based on quantum mechanical path integrals initially did not agree with results known from other methods, that we started a detailed study of path integrals for nonlinear sigma models. These discrepancies have been resolved in the meantime, and the resulting formalism is presented in this book.

The reason that we do not encounter infinities in loop calculations for QM nonlinear sigma models is different from a corresponding statement for QM linear sigma models. For a linear sigma model with a kinetic term $\frac{1}{2}\dot{x}^i\dot{x}^i$ on an infinite t-interval, the propagator behaves as $1/k^2$ for large momenta, and vertices from $V(x)$ do not contain derivatives, hence loops $\int dk[\cdots]$ will always be finite. For nonlinear sigma models with $L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$, propagators still behave like k^{-2} but vertices now behave like k^2 (as in ordinary quantum gravity), hence single loops are linearly divergent by power counting and double loops are logarithmically

[3] Their approach uses a particular linear combination of general coordinate and local Lorentz transformations, and for this symmetry one only needs to evaluate single loops on the worldline. However if one directly computes the anomaly of the Lorentz operator $\gamma^{\mu\nu}\gamma_5$, using the same steps as in the case of the chiral operator γ_5 for gauge fields in flat space, one needs higher loops on the worldline. We discuss this at the end of Section 6.3.