

## Superfractals

*Superfractals* is the long awaited successor to *Fractals Everywhere*, in which the power and beauty of iterated function systems (IFSs) were introduced and applied to the production of startling and original images that reflect complex structures found for example in nature. This provoked the question whether there is a deeper connection between topology, geometry, IFSs and codes on the one hand and biology, DNA and protein development on the other. Now, 20 years later, Professor Barnsley brings the story up to date by explaining how IFSs have developed in order to address this issue. New ideas such as fractal tops and superIFSs are introduced, and the classical deterministic approach is combined with probabilistic ideas to produce new mathematics and algorithms that reveal a theory which could have applications in computer graphics, bioinformatics, economics, signal processing and beyond. For the first time these ideas are explained in book form and illustrated with breathtaking pictures. The text is accessible to all mathematical scientists with some knowledge of calculus and will open up new ways in which the world can be seen.

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MICHAEL FIELDING BARNSELY

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For my daughters, Diana and Rose

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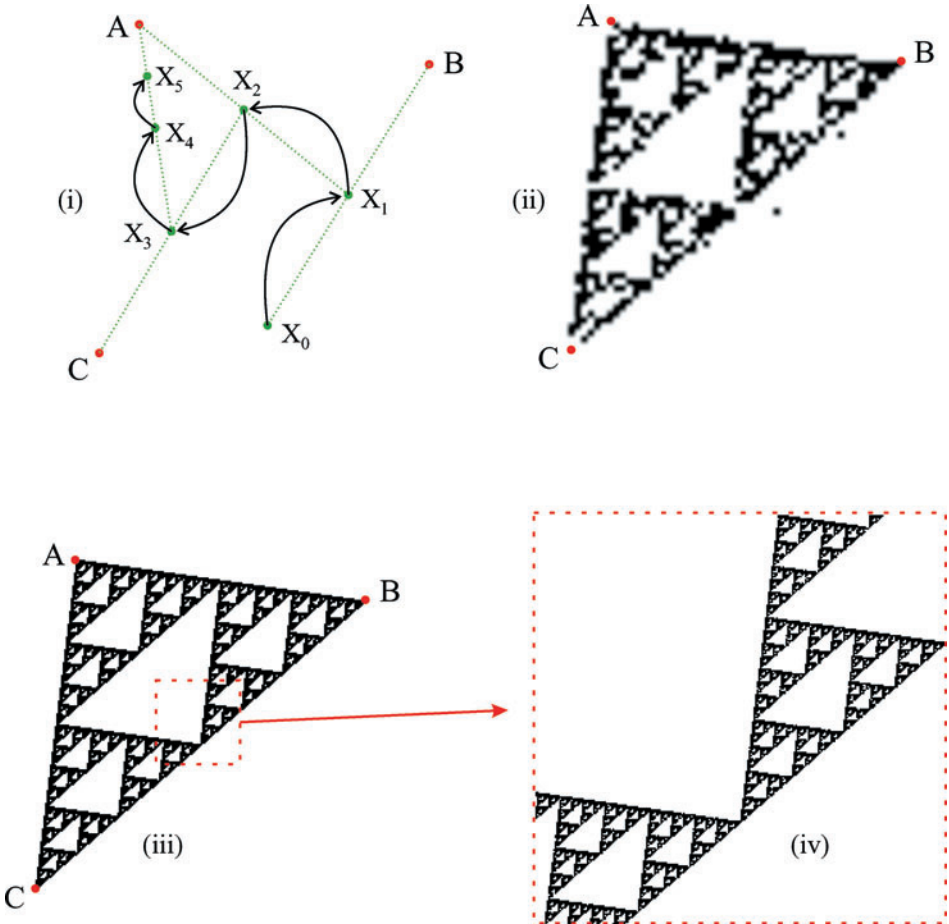
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**Figure 0.1** In the chaos game, one of a few simple rules is selected at random and applied to a point, to yield a new point. This random step is repeated over and over again, to produce an 'attractor'. Here the attractor is a Sierpinski triangle. The figures illustrate (i) the first few points, (ii) the result after 1000 iterations, (iii) the same result at a higher resolution with outliers discarded, (iv) a magnification of (iii). What happens if you change the rules?