INTRODUCTION

0.1 The chaos game

The following process, which I call the ‘chaos game’, provides a simple introduction to the idea of an **iterated function system** (IFS) and its **attractor**.

Mark four points on a sheet of paper. Label three of them **A**, **B** and **C** and label the remaining point $\mathbf{X}_0$, as in Figure 0.1(i). Label two faces of a six-sided die A, two other faces B and the remaining two faces C, or devise your own way of producing a random sequence of the symbols A, B and C.

Roll the die, to choose randomly a symbol A, B or C. On the paper, mark the midpoint between $\mathbf{X}_0$ and the point labelled by the selected symbol. Call this midpoint $\mathbf{X}_1$. For example, if the result of rolling the die is B then $\mathbf{X}_1$ is the midpoint between $\mathbf{X}_0$ and **B**.

Roll the die again. Plot the midpoint between $\mathbf{X}_1$ and the point whose label shows on the die. Call this new point $\mathbf{X}_2$. You get the idea. Roll the die again, and again, . . . , and plot a new midpoint on the paper each time. The result, on the sheet of paper, is very likely to look something like Figure 0.1(ii). It is an approximate picture of a **Sierpinski triangle**, with some extra ‘outlier’ points.

Suppose that you carry out a similar experiment using a computer. Then you can compute accurately a sequence of millions of points

$$\mathbf{X}_0,\quad \mathbf{X}_1,\quad \mathbf{X}_2, \ldots,\quad \mathbf{X}_{10\,000\,000},\quad \mathbf{X}_{10\,000\,001},\quad \ldots$$

and print them out as a high-resolution picture. If the points **A**, **B** and **C** are fixed then each time you run the experiment you are likely to obtain a different picture of the Sierpinski triangle, but only slightly different. In fact, if you work at a resolution of 256×256, compute ten million points and discard the first sixteen points, then it is probable that the resulting picture will look the same each time you run the experiment. An illustration of such a result is shown in Figure 0.1(iii), (iv).

Almost always, regardless of the choice of starting point $\mathbf{X}_0$ and regardless of the particular sequence of random choices, the sequence of points $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ seems to be drawn towards, or ‘attracted to’, the Sierpinski triangle; after sufficiently many random iterations, the successive points appear, at viewing resolution, to lie exactly on the Sierpinski triangle, and to dance around on it forever.

Table 0.1 *Coefficients of the IFS that created Figures 0.2 and 0.3*

n	a_n	b_n	c_n	d_n	e_n	k_n	g_n	h_n	j_n	p_n
1	19.05	0.72	1.86	−0.15	16.9	−0.28	5.63	2.01	20.0	$\frac{60}{100}$
2	0.2	4.4	7.5	−0.3	−4.4	−10.4	0.2	8.8	15.4	$\frac{1}{100}$
3	96.5	35.2	5.8	−131.4	−6.5	19.1	134.8	30.7	7.5	$\frac{20}{100}$
4	−32.5	5.81	−2.9	122.9	−0.1	−19.9	−128.1	−24.3	−5.8	$\frac{19}{100}$

0.2 Attractors of iterated function systems

In the above example the IFS consists of three simple rules, each of which moves the current point to a new location.

Rule 1: Move to the point midway between the current location and **A**.
Rule 2: Move to the point midway between the current location and **B**.
Rule 3: Move to the point midway between the current location and **C**.

We can write these rules in terms of three functions f_1, f_2, f_3 that map the euclidean plane into itself. For example, using coordinate notation, suppose that $\mathbf{A} = (2, 1)$, $\mathbf{B} = (3, 0)$ and $\mathbf{C} = (4, 0)$. Then we define

$$f_1(x, y) = \left(\frac{x+2}{2}, \frac{y+1}{2}\right), \quad f_2(x, y) = \left(\frac{x+3}{2}, \frac{y}{2}\right),$$

$$f_3(x, y) = \left(\frac{x+4}{2}, \frac{y}{2}\right).$$

Using this notation the repeated step in the chaos game can be expressed as

$$(x_{i+1}, y_{i+1}) = f_{\sigma_i}(x_i, y_i) \quad \text{for } i = 0, 1, 2, \ldots$$

where σ_i is a number randomly chosen from the set $\{1, 2, 3\}$ and $\mathbf{X}_i = (x_i, y_i)$. The collection of functions f_1, f_2 and f_3 is called an **iterated function system** (**IFS**). It is denoted by $\{\mathbb{R}^2; f_1, f_2, f_3\}$, where $\mathbb{R}^2$ is the euclidean plane, the space on which the functions act. The Sierpinski triangle is an **attractor** of this IFS.

A different example of an IFS is $\{\square; f_1, f_2, f_3, f_4\}$, where $\square$ is the unit square, defined in Section 1.2, and the functions f_n are given by

$$f_n(x, y) = \left(\frac{a_n x + b_n y + c_n}{g_n x + h_n y + j_n}, \frac{d_n x + e_n y + k_n}{g_n x + h_n y + j_n}\right) \quad \text{for } n = 1, 2, 3, 4.$$

The coefficients are given in Table 0.1. In this case, to implement the chaos game we apply one of the functions f_1, f_2, f_3, f_4 to the current point $\mathbf{X}_i$, to obtain the the next point $\mathbf{X}_{i+1}$ for $i = 1, 2, 3, \ldots$ We apply f_1 with probability p_1, f_2 with probability p_2, f_3 with probability p_3 and f_4 with probability p_4 . For each step,

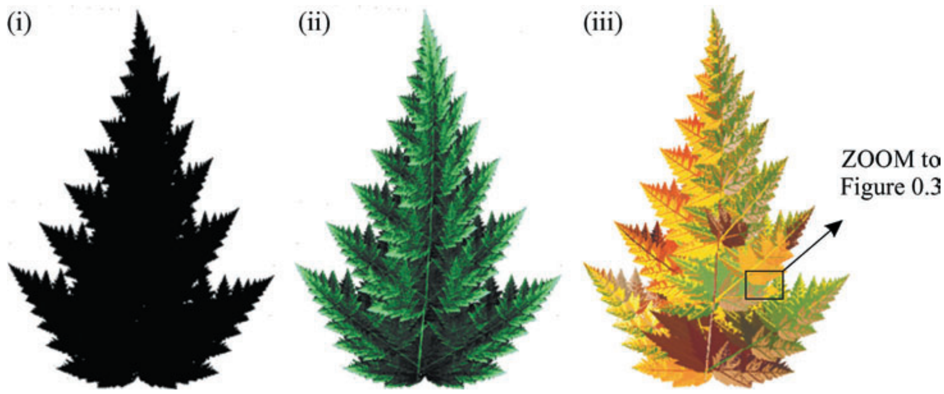

Figure 0.2 Pictures of attractors of an IFS: (i) the set attractor, (ii) the measure attractor and (iii) the fractal top.

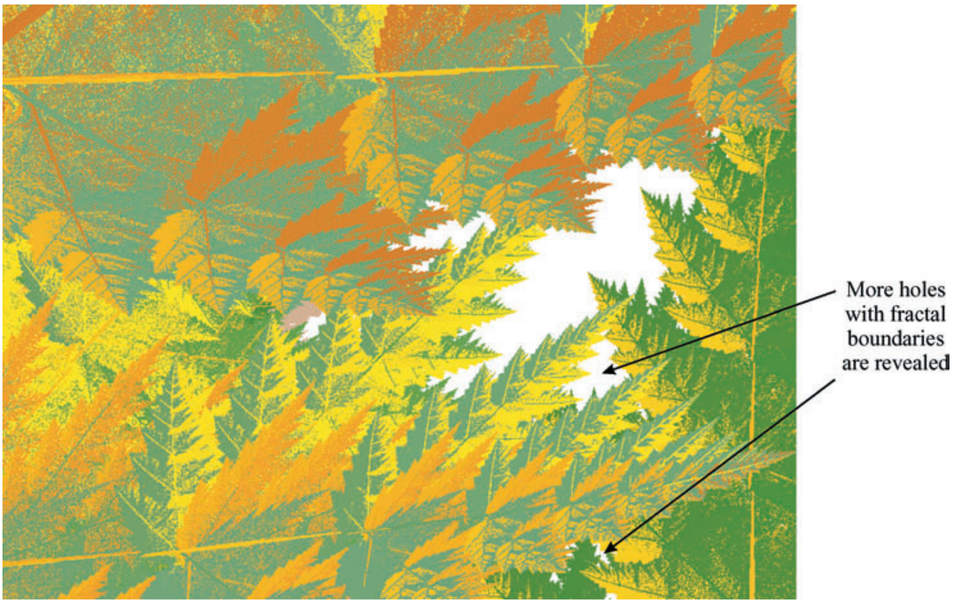

Figure 0.3 Zoom in on the fractal top in Figure 0.2.

the choice of function is made independently of the choices made at all other steps. The probabilities p_n are given in Table 0.1. This time, almost certainly, the sequence of points $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \ldots$ will be attracted to a set that looks like the left-hand picture in Figure 0.2. This is a picture of an attractor of the IFS represented by Table 0.1.

Amazingly, this picture is unlikely to change significantly if the probabilities are adjusted, provided sufficiently many points are plotted. The colours in Figures 0.2(iii) and 0.3 were 'stolen' from Figure 0.4. In Chapter 4 you will discover what this means.

Figure 0.4 Colours were 'stolen' from this picture to produce Figure 0.3 and the image in Figure 0.2(iii).

In this book you will discover different kinds of attractor associated with an IFS. For example, Figure 0.2 illustrates the set attractor, the measure attractor and the fractal top for the IFS in Table 0.1. These beautiful objects may be computed by variants of the chaos game and by other means too. Quite generally, although the IFSs themselves are simple to write down, their attractors are geometrically and topologically complicated. Typically, computer pictures of them can be magnified up endlessly to reveal more and more intricate detail. For example, Figure 0.3 illustrates a tiny hole in the fractal top in Figure 0.2, greatly magnified. Often, simultaneously, such pictures are reminiscent of biological structures and convey the feeling of real-world images, with repetition and disorder combined and the property that one may look ever closer, revealing more and more mysteries. They are suggestive of diverse applications in biology and imaging.

The mathematics in this book is separate from the pictures that illustrate it and the biology that inspired it. Indeed, we will treat all pictures as though they actually are mathematical objects. The attractor of an IFS may be topologically conjugate to a fractal fern without ever leaving the abstract world in which it lives, trapped in mathematical amber, so to speak. All the theorems are independent of the pictures. The mathematics describes something much more general, something bigger, than the pictures.

In this book I try to capture in a precise way a fascinating combination of geometry, topology, probability and pictures. I think that just over the horizon, in the direction in which this book points, there is an unambiguous, new, branch of geometry that combines colour and space. In trying to move towards this goal, I present much new material including the theory of fractal tops, fractal homeomorphisms, orbital pictures and superfractals. At the time of writing only one major paper about superfractals has appeared in print, although a number are in the pipeline.

It is important to read the book from the beginning. Read enough on each page to be sure that you do not miss the themes that build steadily towards two 'peaks' and then the superfractal 'summit'. In Chapter 1 we introduce and explore code space and topology and develop familiarity with metric spaces whose elements are collections of objects. Code space is a major theme of the book. The second major theme, developed in Chapter 2, is elementary transformations and how, specifically and precisely, they act on sets, pictures and measures. Then in Chapter 3 we bring code spaces and transformations together in the framework of IFS semigroups of transformations acting on sets, pictures and measures. It is in the combination of code space and transformations that beautiful new mathematical structures such as orbital pictures, the first 'peak', are discovered.

In Chapter 4 we reach the second 'peak': fractal tops, colour-stealing and fractal homeomorphisms. We discover that we can handle algebraically the topology of some IFS attractors with the same ease that Descartes handled geometrical objects in his Cartesian plane. One application is to computer graphics, via the production of diverse families of beautiful homeomorphisms between images. This chapter combines the chaos game, transformations, identification topologies on code space and basic IFS theory. In effect we study certain limit sets belonging to the objects introduced earlier.

In Chapter 5 we reach the 'summit', which is superfractals. We combine the themes already developed with the concept of V-variability. This enables us to describe and synthesize vast collections of related mathematical objects, be they galleries full of random variations of seascapes or families of related ferns, as illustrated in Figure 0.5. With the aid of our knowledge of transformations, IFS semigroups, code space structure and V-variability we discover that we can produce vast families of homeomorphic objects with random, but not too random, variations. Superfractals provide a bridge, made of IFSs, from deterministic fractals to the world of random fractals. Previously I did not know how to get there.

0.3 Another chaos game

Here is a simple variant of the chaos game. Mark four points on a sheet of paper. Label three of them $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ and label the remaining point $\mathbf{X}_0$. We add two more rules to the three in Section 0.2 above:

Rule 4: *Shift by* $2(\mathbf{B} - \mathbf{C})$.

Rule 5: Rotate by 180° *degrees about the point* $\dfrac{\mathbf{A} + 5\mathbf{B} - 4\mathbf{C}}{2}$.

This time, when you play the game, *remember what the dice showed the last time you rolled it*. Begin by rolling the dice once, to give you a starting value.

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Cambridge University Press
978-0-521-84493-2 - Superfractals
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Figure 0.5 The chaos game produces a sequence of mathematical objects, successively closer and closer to random fractal ferns lying on a 'superfractal'.

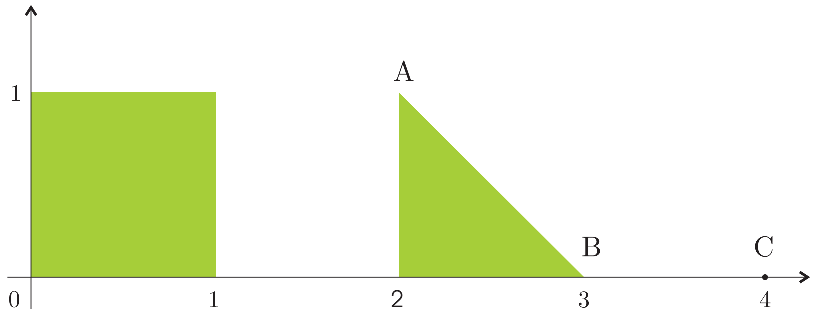

Figure 0.6 The chaos game is played with slightly more complicated rules. The random point now dances on *both* of two classical euclidean objects, a square and a triangle.

Now, each time you roll the dice, if last time it showed A or B and this time it shows A then apply *Rule 1*. If it shows B then apply *Rule 2* but if it shows C then apply *Rule 4*. If last time it showed C, however, and this time it produces A or B then apply *Rule 5* but if it produces C then apply *Rule* 3.

In this new game, if you discard the initial few points then you will obtain a result quite as astonishing as the Sierpinski triangle; you will obtain *simultaneously* two classical geometrical objects, a filled parallelogram and a triangle. See Figure 0.6, which shows the resulting picture when $\mathbf{A} = (2, 1)$, $\mathbf{B} = (3, 0)$ and $\mathbf{C} = (4, 0)$. By following the rules above, the current point will continually dance on the square and the triangle, sometimes moving from one to the other, sometimes moving back again to the first – forever.

So you see, Diana, Rose and gentle reader, this book is about much more than basic fractal geometry. It is about extraordinary transformations of pictures, homeomorphisms between complicated objects and the magic of code space. It is about superfractals.

CHAPTER 1

Codes, metrics and topologies

1.1 Introduction

Any picture may be conceived as a mathematical object, lying on part of the euclidean plane, each point having its own colour. Then it is a strange and wonderful entity. It is mysterious, for you probably cannot see it. And worse, you cannot even describe it in the type of language with which you normally talk about objects you can see; at least, not without making a lot of assumptions. But we want to be able to see, to describe and to make pictures on paper of fractals and other mathematical objects that we feel ought to be capable of representation as pictures. We want to make mathematical models for real-world images, biological entities such as leaves and many other types of data. To be able to do this we need certain parts of the language of mathematics, related to set theory, metric spaces and topology.

Code space There is a remarkable set, called a code space, which consists of an uncountable infinity of points and which can be embedded in the tiniest real interval. A code space can be reorganized in an endless variety of amazing geometrical, topological, ways, to form sets that look like leaves, ferns, cells, flowers and so on. For this reason we think of a code space as being somehow protoplasmic, plastic, impressionable and capable of diverse re-expressions, like the meristem of a plant; see Figure 1.1. This idea is a theme of this chapter and of the whole book.

Structure and contents of this chapter

In this chapter we introduce and discuss spaces, with the focus on those that we will be using later. They include the euclidean plane, code spaces and spaces whose points are certain subsets of other spaces. In particular, we discuss spaces that consist of infinitely many points, such as the real interval [0, 1] and the euclidean plane $\mathbb{R}^2$. We also introduce notation that we shall use throughout the book.

A space may have one or more of the following three properties: (i) its points are organized by means of a system of **addresses** or coordinates; (ii) the relationship between the points of the space is described by means of a **metric** or distance function; (iii) the relationship between the points of the space is described by

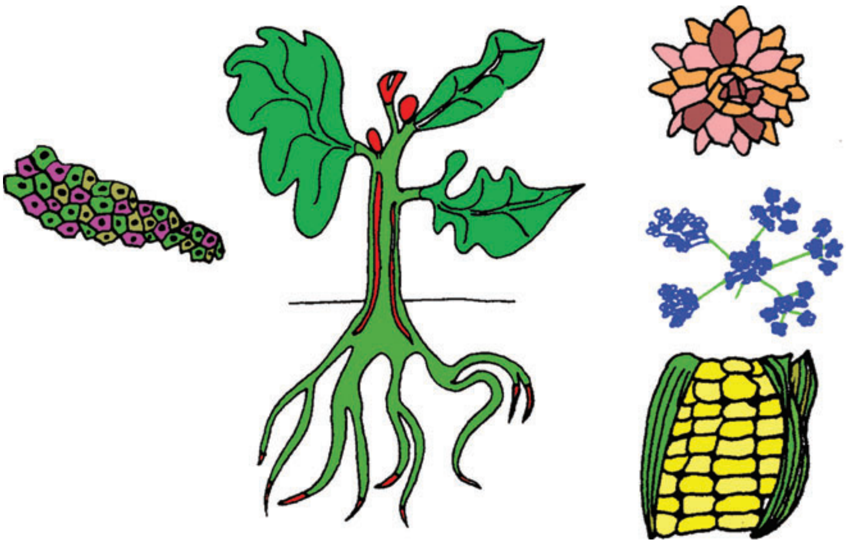

Figure 1.1 '**Meristem**, a specialized section of plant tissue characterized by cell division and growth ... In one type of lateral meristem, called cambium, or vascular cambium, the cells divide and differentiate to form the conducting tissues of the plant, i.e. the wood or xylem, and the phloem.' (*Columbia Encyclopedia*, sixth edition, 2004)

means of a **topology**, with certain subsets labelled 'open'. Typically properties (i), (ii) and (iii) are not independent. Moreover different systems of addresses, diverse metrics and various topologies may be possible on the same space.

We discuss addressing schemes and spaces of addresses, namely code spaces, in Section 1.4. In particular, we explain how addresses for the points in a line segment in $\mathbb{R}^2$ may be defined geometrically via successive bisections. In Section 1.6 we show how diverse metrics may be defined on a code space by embedding it in a space such as $\mathbb{R}^2$. We treat code spaces as very important because of their central role in describing fractals, fractal tops and superfractals in later chapters.

We introduce metric spaces in Section 1.5 and topological spaces in Section 1.8. In Section 1.9 we describe a number of basic, readily constructed, topologies, including identification topologies. An identification topology on a space may be obtained by treating some pairs of points as single points, that is, by 'gluing them together'. In this manner a code space may be given the topology of a line segment, a Möbius strip or a fractal tree. Identification topologies play a very important role in Chapter 4, where we discuss fractal homeomorphisms.

The possible organizational schemes (i), (ii) and (iii) are brought to life by transformations, introduced in Section 1.3. Some of the fundamental properties of a space are those that are preserved by rich collections of transformations such as addressing functions, metric transformations, continuous transformations and homeomorphisms. From this point of view we discuss properties of metric spaces in Section 1.7 and those of topological spaces in Section 1.10. The properties of completeness, defined in Section 1.7, and compactness, defined in Section 1.11, are needed to establish the existence of fractal objects. We describe the conditions under which these properties occur.

Over and above the themes of code spaces, properties of spaces that are preserved under transformations and the nature of euclidean space, a central focus of this chapter, which will carry on throughout the book, is the idea that *the points in a space may themselves be mathematical objects*. For example, they may be mathematical pictures, or measures, defined in Chapter 2. Or they may simply be the nonempty compact subsets of another underlying space.

Thus, the points of a space H_X may be constructed using sets of the points of an underlying space X. Organizational principles such as addresses, metrics and topologies may be inherited from X and provide structure to H_X. Properties of the underlying space X such as compactness and completeness may also be inherited by the space H_X. Moreover, transformations acting on X may be used to define transformations on H_X. These inheritances are important because they enable us to establish the existence of diverse types of fractal in later chapters.

For example, in Section 1.13 we show that the property of being a compact metric space may be inherited from X by a certain space $\mathbb{H}(X)$. The inherited metric, the Hausdorff metric, is discussed earlier, in Section 1.12, with a view to developing our intuition about how it works. This remarkable inheritance continues from generation to generation, from X to $\mathbb{H}(X)$ to $\mathbb{H}(\mathbb{H}(X))$ and so on. It enables us to establish the existence of superfractal sets in Chapter 5.

1.2 Points and spaces

In this section we introduce the notation and nomenclature for points, sets and spaces that we shall use throughout the book.

A **space** is a set. The elements of the set are called the **points** of the space. We use the notation $\mathbb{X}$ to denote a space. The expression $x \in \mathbb{X}$ means that x belongs to the set $\mathbb{X}$ or equivalently that x is a point of the space $\mathbb{X}$. Similarly the expression $x, y \in \mathbb{X}$ means that both x and y are points of $\mathbb{X}$. We say that two points $x, y \in \mathbb{X}$ are **distinct** if $x \neq y$, that is, x is not equal to y. When we consider several spaces at once, we may denote them by $\mathbb{X}, \mathbb{Y}, \ldots$ A space may be empty, that is, it may contain no points.

For illustration, some spaces are shown in Figure 1.2. An important example of a space is $\mathbb{R}$, the set of all finite real numbers. A point $x \in \mathbb{R}$ is simply any number, positive or negative, that can be expressed by a decimal expansion, either finite as in $x = 1.5$ or unending as in $x = -7.93121059912791101\cdots$. We can write

$$\mathbb{R} = \{x : -\infty < x < +\infty\}.$$