Part I
General background

Cambridge University Press
978-0-521-84404-8 - Deformation of Earth Materials: An Introduction to the Rheology of Solid Earth
Shun-ichiro Karato
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1 Stress and strain

The concept of stress and strain is key to the understanding of deformation. When a force is applied to a continuum medium, stress is developed inside it. Stress is the force per unit area acting on a given plane along a certain direction. For a given applied force, the stress developed in a material depends on the orientation of the plane considered. Stress can be decomposed into hydrostatic stress (pressure) and deviatoric stress. Plastic deformation (in non-porous materials) occurs due to deviatoric stress. Deformation is characterized by the deformation gradient tensor, which can be decomposed into rigid body rotation and strain. Deformation such as simple shear involves both strain and rigid body rotation and hence is referred to as rotational deformation whereas pure shear or tri-axial compression involves only strain and has no rigid body rotation and hence is referred to as irrotational deformation. In rotational deformation, the principal axes of strain rotate with respect to those of stress whereas they remain parallel in irrotational deformation. Strain can be decomposed into dilatational (volumetric) strain and shear strain. Plastic deformation (in a non-porous material) causes shear strain and not dilatational strain. Both stress and strain are second-rank tensors, and can be characterized by the orientation of the principal axes and the magnitude of the principal stress and strain and both have three invariants that do not depend on the coordinate system chosen.

Key words stress, strain, deformation gradient, vorticity, principal strain, principal stress, invariants of stress, invariants of strain, normal stress, shear stress, Mohr's circle, the Flinn diagram, foliation, lineation, coaxial deformation, non-coaxial deformation.

1.1. Stress

1.1.1. Definition of stress

This chapter provides a brief summary of the basic concept of stress and strain that is relevant to understanding plastic deformation. For a more comprehensive treatment of stress and strain, the reader may consult Malvern (1969), Mase (1970), Means (1976).

In any deformed or deforming continuum material there must be a force inside it. Consider a small block of a deformed material. Forces acting on the material can be classified into two categories, i.e., a short-range force due to atomic interactions and the long-range force due to an external field such as the gravity field. Therefore the forces that act on this small block include (1) short-range forces due to the displacement of atoms within this block, (2) long-range forces such as gravity that act equally on each atom and (3) the forces that act on this block through the surface from the neighboring materials. The (small) displacements of each atom inside this region cause forces to act on surrounding atoms, but by assumption these forces are short range. Therefore one can consider them as forces between a pair of atoms A and B. However, because of Newton's law of action and counter-action, the forces acting between two atoms are anti-symmetric: $f_{AB} = -f_{BA}$ where $f_{AB\ (BA)}$

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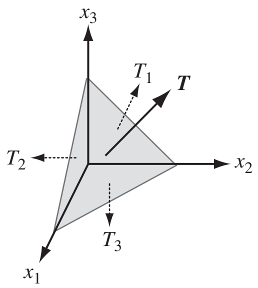

FIGURE 1.1 Forces acting on a small pyramid.

are the force exerted by atom A (B) to B (A). Consequently these forces caused by atomic displacement within a body must cancel. The long-range force is called a *body force*, but if one takes this region as small, then the magnitude of this body force will become negligible compared to the *surface force* (i.e., the third class of force above). Therefore the net force acting on the small region must be the forces across the surface of that region from the neighboring materials. To characterize this force, let us consider a small piece of block that contains a plane with the area of dS and whose normal is $\boldsymbol{n}$ ($\boldsymbol{n}$ is the unit vector). Let $\boldsymbol{T}$ be the force (per unit area) acting on the surface dS from outside this block (positive when the force is compressive) and consider the force balance (Fig. 1.1). The force balance should be attained among the force $\boldsymbol{T}$ as well as the forces $\boldsymbol{T}^{1,2,3}$ that act on the surface $dS_{1,2,3}$ respectively ($dS_{1,2,3}$ are the projected area of dS on the plane normal to the $x_{1,2,3}$ axis). Then the force balance relation for the block yields,

$$\boldsymbol{T}\,dS = \sum_{j=1}^{3} \boldsymbol{T}^j\,dS_j. \tag{1.1}$$

Now using the relation $dS_j = n_j\,dS$, one obtains,

$$T_i = \sum_{j=1}^{3} T_i^j n_j = \sum_{j=1}^{3} \sigma_{ij} n_j \tag{1.2}$$

where T_i is the ith component of the force $\boldsymbol{T}$ and σ_{ij} is the ith component of the traction $\boldsymbol{T}^j$, namely the ith component of force acting on a plane whose normal is the jth direction ($n_{ij} = T_i^j$). This is the definition of *stress*. From the balance of torque, one can also show,

$$\sigma_{ij} = \sigma_{ji}. \tag{1.3}$$

The values of stress thus defined depend on the coordinate system chosen. Let us denote quantities in a new coordinate system by a tilda, then the new coordinate and the old coordinate system are related to each other by,

$$\tilde{x}_i = \sum_{j=1}^{3} a_{ij} x_j \tag{1.4}$$

where a_{ij} is the transformation matrix that satisfies the orthonormality relation,

$$\sum_{j=1}^{3} a_{ij} a_{jm} = \delta_{im} \tag{1.5}$$

where δ_{im} is the Kronecker delta ($\delta_{im} = 1$ for $i = m$, $\delta_{im} = 0$ otherwise). Now in this new coordinate system, we may write a relation similar to equation (1.2) as,

$$\tilde{T}_i = \sum_{j=1}^{3} \tilde{\sigma}_{ij} \tilde{n}_j. \tag{1.6}$$

Noting that the traction ($\boldsymbol{T}$) transforms as a vector in the same way as the coordinate system, equation (1.4), we have,

$$\tilde{T}_i = \sum_{j=1}^{3} a_{ij} T_j. \tag{1.7}$$

Inserting equation (1.2), the relation (1.7) becomes,

$$\tilde{T}_i = \sum_{j,k=1}^{3} \sigma_{jk} a_{ij} n_k. \tag{1.8}$$

Now using the orthonormality relation (1.5), one has,

$$n_i = \sum_{j=1}^{3} a_{ji} \tilde{n}_i. \tag{1.9}$$

Inserting this relation into equation (1.8) and comparing the result with equation (1.6), one obtains,[1]

$$\tilde{\sigma}_{ij} = \sum_{k,l=1}^{3} \sigma_{kl} a_{ik} a_{jl}. \tag{1.10}$$

The quantity that follows this transformation law is referred to as *a second rank tensor*.

1.1.2. Principal stress, stress invariants

In any material, there must be a certain orientation of a plane on which the direction of traction ($\boldsymbol{T}$) is normal to it. For that direction of $\boldsymbol{n}$, one can write,

$$T_i = \sigma n_i \tag{1.11}$$

[1] In the matrix notation, $\tilde{\sigma} = A \cdot \sigma \cdot A^T$ where $A = (a_{ij})$ and $A^T = (a_{ji})$.

where σ is a scalar quantity to be determined. From equations (1.11) and (1.2),

$$\sum_{j=1}^{3}(\sigma_{ij} - \sigma\delta_{ij})n_j = 0. \tag{1.12}$$

For this equation to have a non-trivial solution other than $\boldsymbol{n} = 0$, one must have,

$$\left|\sigma_{ij} - \sigma\delta_{ij}\right| = 0 \tag{1.13}$$

where $|X_{ij}|$ is the determinant of a matrix X_{ij}. Writing equation (1.13) explicitly, one obtains,

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \sigma \end{vmatrix} = -\sigma^3 + I_\sigma\sigma^2 + II_\sigma\sigma + III_\sigma = 0 \tag{1.14}$$

with

$$I_\sigma = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{1.15a}$$

$$II_\sigma = -\sigma_{11}\sigma_{22} - \sigma_{11}\sigma_{33} - \sigma_{33}\sigma_{22} + \sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2 \tag{1.15b}$$

$$III_\sigma = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2. \tag{1.15c}$$

Therefore, there are three solutions to equation (1.14), $\sigma_1, \sigma_2, \sigma_3 (\sigma_1 > \sigma_2 > \sigma_3)$.These are referred to as the *principal stresses*. The corresponding $\boldsymbol{n}$ is *the orientation of principal stress*. If the stress tensor is written using the coordinate whose orientation coincides with the orientation of principal stress, then,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}. \tag{1.16}$$

It is also seen that because equation (1.14) is a scalar equation, the values of I_σ, II_σ and III_σ are independent of the coordinate. These quantities are called the *invariants of stress tensor*. These quantities play important roles in the formal theory of plasticity (see Section 3.3). Equations (1.15a–c) can also be written in terms of the principal stress as,

$$I_\sigma = \sigma_1 + \sigma_2 + \sigma_3 \tag{1.17a}$$

$$II_\sigma = -\sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 \tag{1.17b}$$

and

$$III_\sigma = \sigma_1\sigma_2\sigma_3. \tag{1.17c}$$

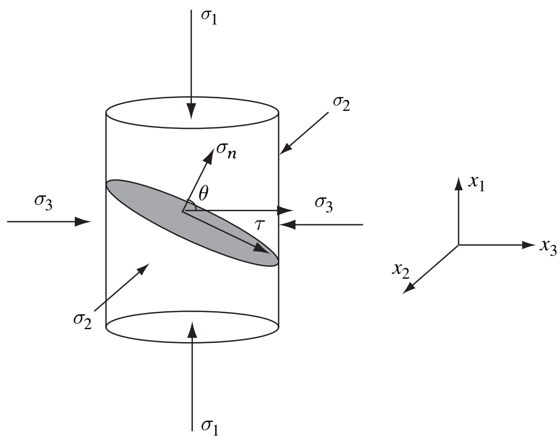

FIGURE 1.2 Geometry of normal and shear stress on a plane.

1.1.3. Normal stress, shear stress, Mohr's circle

Now let us consider the normal and shear stress on a given plane subjected to an external force (Fig. 1.2). Let x_1 be the axis parallel to the maximum compressional stress σ_1 and x_2 and x_3 be the axes perpendicular to x_1. Consider a plane whose normal is at the angle θ from x_3 (positive counterclockwise). Now, we define a new coordinate system whose x_1' axis is normal to the plane, but the x_2' axis is the same as the x_2 axis. Then the transformation matrix is,

$$[a_{ij}] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.18}$$

and hence,

$$[\tilde{\sigma}_{ij}] = \begin{bmatrix} \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\cos 2\theta & 0 & \frac{\sigma_1-\sigma_3}{2}\sin 2\theta \\ 0 & \sigma_2 & 0 \\ \frac{\sigma_1-\sigma_3}{2}\sin 2\theta & 0 & \frac{\sigma_1+\sigma_3}{2} - \frac{\sigma_1-\sigma_3}{2}\cos 2\theta \end{bmatrix}. \tag{1.19}$$

Problem 1.1

Derive equation (1.19).

Solution

The stress tensor (1.16) can be rotated through the operation of the transformation matrix (1.18) using equation (1.10),

$$[\tilde{\sigma}_{ij}] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sigma_1+\sigma_3}{2} + \frac{\sigma_1-\sigma_3}{2}\cos 2\theta & 0 & \frac{\sigma_1-\sigma_3}{2}\sin 2\theta \\ 0 & \sigma_2 & 0 \\ \frac{\sigma_1-\sigma_3}{2}\sin 2\theta & 0 & \frac{\sigma_1+\sigma_3}{2} - \frac{\sigma_1-\sigma_3}{2}\cos 2\theta \end{bmatrix}.$$

Therefore the shear stress τ and normal stress σ_n on this plane are

$$\tilde{\sigma}_{13} \equiv \tau = \frac{\sigma_1 - \sigma_3}{2}\sin 2\theta \quad (1.20)$$

and

$$\tilde{\sigma}_{33} \equiv \sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2}\cos 2\theta \quad (1.21)$$

respectively. It follows that the maximum shear stress is on the two conjugate planes that are inclined by $\pm\pi/4$ with respect to the x_1 axis and its absolute magnitude is $(\sigma_1 - \sigma_3)/2$. Similarly, the maximum compressional stress is on a plane that is normal to the x_1 axis and its value is σ_1. It is customary to use $\sigma_1 - \sigma_3$ as (differential (or deviatoric)) stress in rock deformation literature, but the shear stress, $\tau \equiv (\sigma_1 - \sigma_3)/2$, is also often used. Eliminating θ from equations (1.20) and (1.21), one has,

$$\tau^2 + \left(\sigma_n - \frac{\sigma_1+\sigma_3}{2}\right)^2 = \frac{1}{4}(\sigma_1 - \sigma_3)^2. \quad (1.22)$$

Thus, the normal and shear stress on planes with various orientations can be visualized on a two-dimensional plane (τ–σ_n space) as a circle whose center is located at $(0, (\sigma_1 + \sigma_3)/2)$ and the radius $(\sigma_1 - \sigma_3)/2$ (Fig. 1.3). This is called a *Mohr's circle* and plays an important role in studying the brittle fracture that is controlled by the stress state (shear–normal stress ratio; see Section 7.3).

When $\sigma_1 = \sigma_2 = \sigma_3 (= P)$, then the stress is isotropic (hydrostatic). The hydrostatic component of stress does not cause plastic flow (this is not true for porous materials, but we do not discuss porous materials here), so it is useful to define *deviatoric stress*

$$\sigma'_{ij} \equiv \sigma_{ij} - \delta_{ij} P. \quad (1.23)$$

When we discuss plastic deformation in this book, we use σ_{ij} (without prime) to mean deviatoric stress for simplicity.

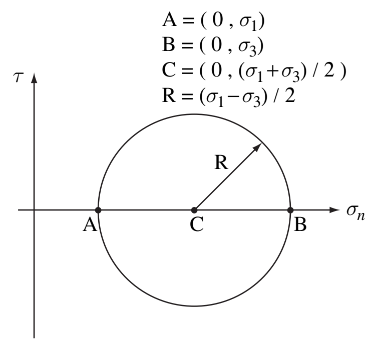

FIGURE 1.3 A Mohr circle corresponding to two-dimensional stress showing the variation of normal, σ_n, and shear stress, τ, on a plane.

Problem 1.2

Show that the second invariant of deviatoric stress can be written as $II_{\sigma'} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$.

Solution

If one uses a coordinate system parallel to the principal axes of stress, from equation (1.15), one has $II_{\sigma'} = -\sigma'_1\sigma'_2 - \sigma'_1\sigma'_3 - \sigma'_3\sigma'_2$. Using $I_{\sigma'} = \sigma'_1 + \sigma'_2 + \sigma'_3 = 0$, one finds $I^2_{\sigma} = \sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3 + 2(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1) = 0$. Therefore $II_{\sigma'} = \frac{1}{2}(\sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3)$. Now, inserting $\sigma'_1 = \sigma_1 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ etc., one obtains $II_{\sigma'} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$.

Problem 1.3

Show that when the stress has axial symmetry with respect to the x_1 axis (i.e., $\sigma_2 = \sigma_3$), then $\sigma_n = P + (\sigma_1 - \sigma_3)(\cos^2\theta - \frac{1}{3})$.

Solution

From (1.21), one obtains, $\sigma_n = (\sigma_1 + \sigma_3)/2 + ((\sigma_1 - \sigma_3)/2)\cos 2\theta$. Now $\cos 2\theta = 2\cos^2\theta - 1$ and $P = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\sigma_1 + 2\sigma_3) = \sigma_1 - \frac{2}{3}(\sigma_1 - \sigma_3)$. Therefore $\sigma_n = P + (\sigma_1 - \sigma_3)(\cos^2\theta - \frac{1}{3})$.

Equations similar to (1.15)–(1.17) apply to the deviatoric stress.

1.2. Deformation, strain

1.2.1. Definition of strain

Deformation refers to a change in the shape of a material. Since homogeneous displacement of material points does not cause deformation, deformation must be related to *spatial variation* or *gradient* of displacement. Therefore, deformation is characterized by a displacement gradient tensor,

$$d_{ij} \equiv \frac{\partial u_i}{\partial x_j}. \tag{1.24}$$

where u_i is the displacement and x_j is the spatial coordinate (after deformation). However, this displacement gradient includes the rigid-body rotation that has nothing to do with deformation. In order to focus on deformation, let us consider two adjacent material points $P_0(\boldsymbol{X})$ and $Q_0(\boldsymbol{X}+d\boldsymbol{X})$, which will be moved to $P(\boldsymbol{x})$ and $Q(\boldsymbol{x}+d\boldsymbol{x})$ after deformation (Fig. 1.4). A small vector connecting P_0 and Q_0, $d\boldsymbol{X}$, changes to $d\boldsymbol{x}$ after deformation. Let us consider how the length of these two segments changes. The difference in the squares of the length of these small elements is given by,

$$\begin{aligned}(dx)^2-(dX)^2 &= \sum_{i=1}^{3}(dx_i)^2-\sum_{i=1}^{3}(dX_i)^2\\ &= \sum_{i,j,k=1}^{3}\left(\delta_{ij}-\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right)dx_i\,dx_j.\end{aligned} \tag{1.25}$$

Therefore deformation is characterized by a quantity,

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\delta_{ij}-\sum_{k=1}^{3}\frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \tag{1.26}$$

which is the definition of *strain*, ε_{ij}. With this definition, the equation (1.25) can be written as,

$$(dx)^2-(dX)^2 \equiv 2\sum_{i,j}\varepsilon_{ij}\,dx_i\,dx_j. \tag{1.27}$$

From the definition of strain, it immediately follows that the strain is a symmetric tensor, namely,

$$\varepsilon_{ij}=\varepsilon_{ji}. \tag{1.28}$$

Now, from Fig. 1.4, one obtains,

$$du_i = dx_i - dX_i \tag{1.29}$$

hence

$$\frac{\partial u_i}{\partial x_j}=\delta_{ij}-\frac{\partial X_i}{\partial x_j}. \tag{1.30}$$

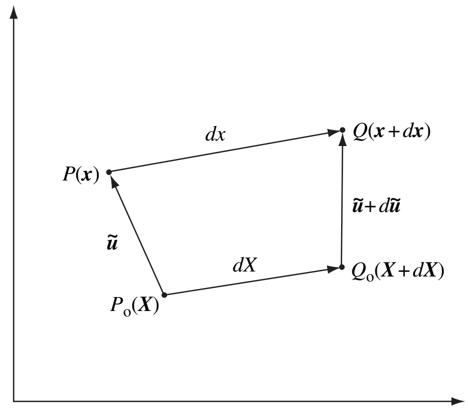

FIGURE 1.4 Deformation causes the change in relative positions of material points.

Inserting equation (1.30) into (1.26) one finds,

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\sum_{k=1}^{3}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right). \tag{1.31}$$

This definition of strain uses the deformed state as a reference frame and is called the *Eulerian strain*. One can also define strain using the initial, undeformed reference state. This is referred to as the *Lagrangian strain*. For small strain, there is no difference between the Eulerian and Lagrangian strain and both are reduced to[2]

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right). \tag{1.32}$$

1.2.2. Meaning of strain tensor

The interpretation of strain is easier in this linearized form. The displacement gradient can be decomposed into two components,

$$\frac{\partial u_i}{\partial x_j}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right). \tag{1.33}$$

The first component is a symmetric part,

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\varepsilon_{ji} \tag{1.34}$$

which represents the strain (as will be shown later in this chapter).

[2] Note that in some literature, another definition of shear strain is used in which $\varepsilon_{ij}=\partial u_i/\partial x_j+\partial u_j/\partial x_i$ for $i\neq j$ and $\varepsilon_{ii}=\partial u_i/\partial x_i$; e.g., Hobbs *et al.* (1976). In such a case, the symbol γ_{ij} is often used for the non-diagonal ($i\neq j$) strain component instead of ε_{ij}.

Let us first consider the physical meaning of the second part, $\frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})$. The second part is an anti-symmetric tensor, namely,

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\omega_{ji}\,(\omega_{ii} = 0). \tag{1.35}$$

The displacement of a small vector du_j due to the operation of this matrix is given by,

$$d\tilde{u}_i^{\omega} = \sum_{j=1}^{3} \omega_{ij}\, du_j. \tag{1.36}$$

Since $\omega_{ii} = 0$, the displacement occurs only to the directions that are normal to the initial orientation. Therefore the operation of this matrix causes the rotation of material points with the axis that is normal to both ith and jth directions with the magnitude (positive clockwise),

$$\tan\theta_{ij} = -\frac{d\tilde{u}_i^{\omega}}{du_i} = -\omega_{ji} = \omega_{ij}. \tag{1.37}$$

(Again this rotation tensor is defined using the deformed state. So it is referred to as the *Eulerian rotation tensor*.) To represent this, a rotation vector is often used that is defined as,

$$\boldsymbol{\omega}(= (\omega_1, \omega_2, \omega_3)) \equiv (\omega_{23}, \omega_{31}, \omega_{12}). \tag{1.38}$$

Thus ω_i represents a rotation with respect to the ith axis. The anti-symmetric tensor, ω_{ij}, is often referred to as a *vorticity* tensor.

Now we turn to the symmetric part of displacement gradient tensor, ε_{ij}. The displacement due to the operation of ε_{ij} is,

$$d\tilde{u}_i^{\varepsilon} = \sum_{j=1}^{3} \varepsilon_{ij}\, du_j. \tag{1.39}$$

From equation (1.39), it follows that the length of a component of vector u_i^0 changes to,

$$\tilde{u}_i = (1 + \varepsilon_{ii})u_i^0. \tag{1.40}$$

Therefore the diagonal component of strain tensor represents the change in length, so that this component of strain, ε_{ii}, is called *normal strain*. Consequently,

$$\frac{V}{V_0} = (1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) \approx 1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{1.41}$$

where V_0 is initial volume and V is the final volume and the strain is assumed to be small (this assumption can be relaxed and the same argument can be applied to a finite strain, see e.g., Mase (1970)). Thus,

$$\sum_{k=1}^{3} \varepsilon_{kk} = \frac{\triangle V}{V}. \tag{1.42}$$

Obviously, normal strain can be present in deformation without a volume change. For example, $\varepsilon_{ij} = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & -\frac{1}{2}\varepsilon & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon \end{pmatrix}$ represents an elongation along the 1-axis and contraction along the 2 and 3 axes without volume change.

Now let us consider the off-diagonal components of strain tensor. From equation (1.39), it is clear that when all the diagonal components are zero, then all the displacement vectors must be normal to the direction of the initial vector. Therefore, there is no change in length due to the off-diagonal component of strain. Note, also, that since strain is a symmetric tensor, $\varepsilon_{ij} = \varepsilon_{ji}$, the directions of rotation of two orthogonal axes are toward the opposite direction with the same magnitude (Fig. 1.5). Consequently, the angle of two orthogonal axes change from $\pi/2$ to (see Problem 1.4),

$$\frac{\pi}{2} - \tan^{-1} 2\varepsilon_{ij}. \tag{1.43}$$

Therefore, the off-diagonal components of strain tensor (i.e., ε_{ij} with $i \neq j$) represent the shape change without volume change, namely shear strain.

Problem 1.4*

Derive equation (1.43). (Assume a small strain for simplicity. The result also works for a finite strain, see Mase (1970).)

Solution

Let the small angle of rotation of the i axis to the j axis due to the operation of strain tensor be $\delta\theta_{ij}$ (positive clockwise), then (Fig. 1.5),

$$\tan\delta\theta_{ij} = -\frac{d\tilde{u}_j}{du_i} \approx \delta\theta_{ij} = -(\varepsilon_{ji} + \omega_{ji}) = -\varepsilon_{ij} + \omega_{ij}.$$

Similarly, if the rotation of the j axis relative to the i axis is $\delta\theta_{ji}$, one obtains,

$$\tan\delta\theta_{ji} = -\frac{d\tilde{u}_i}{du_j} \approx \delta\theta_{ji} = -(\varepsilon_{ij} + \omega_{ij}) = -\varepsilon_{ij} - \omega_{ij}.$$

(Note that the rigid-body rotations of the two axes are opposite with the same magnitude.) Therefore, the net change in the angle between i and j axes is given by

$$\triangle\theta_{ij} = \delta\theta_{ij} + \delta\theta_{ji} = -2\varepsilon_{ij} \sim \tan\triangle\theta_{ij}.$$

Hence $\triangle\theta_{ij} = -\tan^{-1} 2\varepsilon_{ij}$.

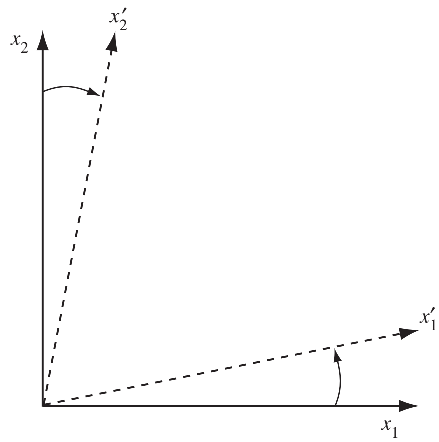

FIGURE 1.5 Geometry of shear deformation.

1.2.3. Principal strain, strain ellipsoid

We have seen two different cases for strain, one in which the displacement caused by the strain tensor is normal to the original direction of the material line and another where the displacement is normal to the original material line. In this section, we will learn that in any material and in any geometry of strain, there are three directions along which the displacement is normal to the direction of original line segment. These are referred to as the orientation of principal strain, and the magnitude of strain along these orientations are called principal strain.

One can define the principal strains ($\varepsilon_1, \varepsilon_2, \varepsilon_3$; $\varepsilon_1 > \varepsilon_2 > \varepsilon_3$) in the following way. Recall that the normal displacement along the direction i, $\delta\tilde{u}_i$, along the vector $\boldsymbol{u}$ is given by,

$$\delta\tilde{u}_i = \sum_{j=1}^{3} \varepsilon_{ij} u_j. \tag{1.44}$$

Now, let $\boldsymbol{u}$ be the direction in space along which the displacement is parallel to the direction $\boldsymbol{u}$. Then,

$$\delta\tilde{u}_i = \varepsilon u_i \tag{1.45}$$

where ε is a scalar quantity to be determined. From equations (1.44) and (1.45),

$$\sum_{j=1}^{3} (\varepsilon_{ij} - \varepsilon\delta_{ij}) u_j = 0. \tag{1.46}$$

For this equation to have a non-trivial solution other than $\boldsymbol{u} = 0$, one must have,

$$|\varepsilon_{ij} - \varepsilon\delta_{ij}| = 0 \tag{1.47}$$

where $|X_{ij}|$ is the determinant of a matrix X_{ij}. Writing equation (1.47) explicitly, one gets,

$$\begin{vmatrix} \varepsilon_{11} - \varepsilon & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon & \varepsilon_{32} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon \end{vmatrix} = -\varepsilon^3 + I_\varepsilon \varepsilon^2 + II_\varepsilon \varepsilon + III_\varepsilon = 0 \tag{1.48}$$

with

$$I_\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{1.49a}$$

$$II_\varepsilon = -\varepsilon_{11}\varepsilon_{22} - \varepsilon_{11}\varepsilon_{33} - \varepsilon_{33}\varepsilon_{22} + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 \tag{1.49b}$$

$$III_\varepsilon = \varepsilon_{11}\varepsilon_{22}\varepsilon_{33} + 2\varepsilon_{12}\varepsilon_{23}\varepsilon_{31} - \varepsilon_{11}\varepsilon_{23}^2 - \varepsilon_{22}\varepsilon_{13}^2 - \varepsilon_{33}\varepsilon_{12}^2. \tag{1.49c}$$

Therefore, there are three solutions of equation (1.48), $\varepsilon_1, \varepsilon_2, \varepsilon_3 (\varepsilon_1 > \varepsilon_2 > \varepsilon_3)$. These are referred to as *the principal strain*. The corresponding $\boldsymbol{u}^0$ are *the orientations of principal strain*. If the strain tensor is written using the coordinate whose orientation coincides with the orientation of principal strain, then,

$$[\varepsilon_{ij}] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{1.50}$$

A strain ellipsoid is a useful way to visualize the geometry of strain. Let us consider a spherical body in a space and deform it. The shape of a sphere is described by,

$$(u_1)^2 + (u_2)^2 + (u_3)^2 = 1. \tag{1.51}$$

The shape of the sphere will change due to deformation. Let us choose a coordinate system such that the directions of 1, 2 and 3 axes coincide with the directions of principal strain. Then the length of each axis of the original sphere along each direction of the coordinate system should change to $\tilde{u}_i = (1 + \varepsilon_{ii})u_i$, and therefore the sphere will change to an ellipsoid,

$$\frac{(\tilde{u}_1)^2}{(1+\varepsilon_1)^2} + \frac{(\tilde{u}_2)^2}{(1+\varepsilon_2)^2} + \frac{(\tilde{u}_3)^2}{(1+\varepsilon_3)^2} = 1. \tag{1.52}$$

A three-dimensional ellipsoid defined by this equation is called *a strain ellipsoid*. For example, if the shape of grains is initially spherical, then the shape of grains after deformation represents the strain ellipsoid. The strain of a rock specimen can be determined by the measurements of the shape of grains or some objects whose initial shape is inferred to be nearly spherical.

Problem 1.5*

Consider a simple shear deformation in which the displacement of material occurs only in one direction (the displacement vector is given by $\boldsymbol{u} = (\gamma y,\ 0,\ 0)$). Calculate the strain ellipsoid, and find how the principal axes of the strain ellipsoid rotate with strain. Also find the relation between the angle of tilt of the initially vertical line and the angle of the maximum elongation direction relative to the horizontal axis.

Solution

For simplicity, let us analyze the geometry in the x–y plane (normal to the shear plane) where shear occurs. Consider a circle defined by $x^2 + y^2 = 1$. By deformation, this circle changes to an ellipsoid, $(x + \gamma y)^2 + y^2 = 1$, i.e.,

$$x^2 + 2\gamma xy + (\gamma^2 + 1)y^2 = 1. \quad (1)$$

Now let us find a new coordinate system that is tilted from the original one by an angle θ (positive counter-clockwise). With this new coordinate system, $(x, y) \rightarrow (X, Y)$ with

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \quad (2)$$

By inserting this relation into (1), one finds,

$$A_{XX}X^2 + A_{XY}XY + A_{YY}Y^2 = 1 \quad (3)$$

with

$$\begin{pmatrix} A_{XX} \\ A_{XY} \\ A_{YY} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma^2\cos 2\theta - \gamma\sin 2\theta \\ 2\gamma(\cos 2\theta - \frac{1}{2}\gamma\sin 2\theta) \\ 1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\gamma^2\cos 2\theta + \gamma\sin 2\theta \end{pmatrix} \quad (4)$$

Now, in order to obtain the orientation in which the X–Y directions coincide with the orientations of principal strain, we set $A_{XY} = 0$, and get $\tan 2\theta = 2/\gamma$. $A_{XX} < A_{YY}$ and therefore X is the direction of maximum elongation. Because the change in the angle (φ) of the initially vertical line from the vertical direction is determined by the strain as $\tan\varphi = \gamma$, we find,

$$\begin{aligned} \tan\theta &= \frac{1}{2}\left(-\gamma + \sqrt{4 + \gamma^2}\right) \\ &= \frac{1}{2}\left(-\tan\varphi + \sqrt{4 + \tan^2\varphi}\right) \end{aligned}. \quad (5)$$

At $\gamma = 0, \theta = \pi/4$. As strain goes to infinity, $\gamma \rightarrow \infty$, i.e., $\varphi \rightarrow \pi/2$, and $\tan\theta \rightarrow 0$ hence $\theta \rightarrow 0$: the direction of maximum elongation approaches the direction of shear. $\varepsilon_1 = A_{XX}^{-1/2} - 1$ changes from 0 at $\gamma = 0$ to ∞ as $\gamma \rightarrow \infty$ and $\varepsilon_2 = A_{yy}^{-1/2} - 1$ changes from 0 at $\gamma = 0$ to –1 at $\gamma \rightarrow \infty$.

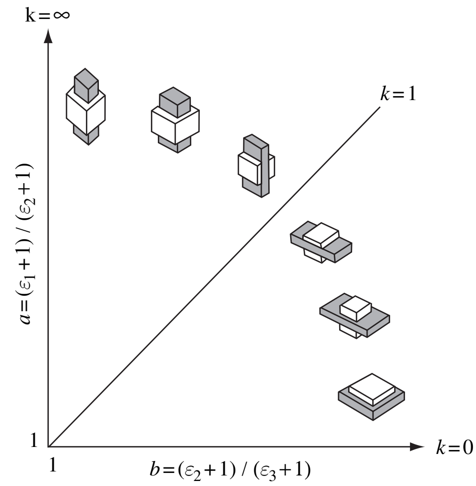

FIGURE 1.6 The Flinn diagram (after HOBBS *et al.*, 1976).

1.2.4. The Flinn diagram

The three principal strains define the geometry of the strain ellipsoid. Consequently, the shape of the strain ellipsoid is completely characterized by two ratios, $a \equiv (\varepsilon_1 + 1)/(\varepsilon_2 + 1)$ and $b \equiv (\varepsilon_2 + 1)/(\varepsilon_3 + 1)$. A diagram showing strain geometry on an a–b plane is called *the Flinn diagram* (Fig. 1.6) (FLINN, 1962). In this diagram, for points along the horizontal axis, $k \equiv (a - 1)/(b - 1) = 0$, and they correspond to the flattening strain ($\varepsilon_1 = \varepsilon_2 > \varepsilon_3 (a = 1, b > 1)$). For points along the vertical axis, $k = \infty$, and they correspond to the extensional strain ($\varepsilon_1 > \varepsilon_2 = \varepsilon_3 (b = 1, a > 1)$). For points along the central line, $k = 1$ ($a = b$, i.e., $(\varepsilon_1 + 1)/(\varepsilon_2 + 1) = (\varepsilon_2 + 1)/(\varepsilon_3 + 1)$) and deformation is plane strain (two-dimensional strain where $\varepsilon_2 = 0$), when there is no volume change during deformation (see Problem 1.6).

Problem 1.6

Show that the deformation of materials represented by the points on the line for $k = 1$ in the Flinn diagram is plane strain (two-dimensional strain) if the volume is conserved.

Solution

If the volume is conserved by deformation, then $(\varepsilon_1 + 1)(\varepsilon_2 + 1)(\varepsilon_3 + 1) = 1$ (see equation (1.41)).

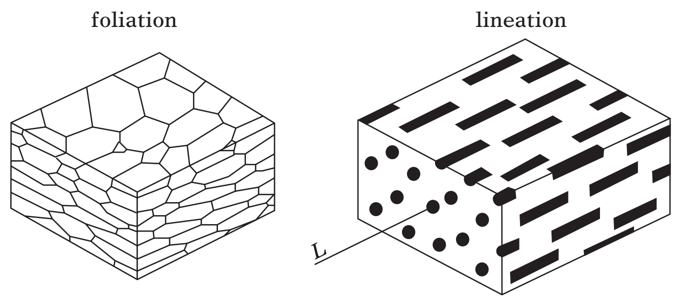

FIGURE 1.7 Typical cases of (a) foliation and (b) lineation.

Combined with the relation $(\varepsilon_1+1)/(\varepsilon_2+1)=(\varepsilon_2+1)/(\varepsilon_3+1)$, we obtain $(\varepsilon_2+1)^3=1$ and hence $\varepsilon_2=0$. Therefore deformation is plane strain.

1.2.5. Foliation, lineation (Fig. 1.7)

When the anisotropic microstructure of a rock is studied, it is critical to define the reference frame of the coordinate. Once one identifies a plane of reference and the reference direction on that plane, then the three orthogonal axes (parallel to lineation (X direction), normal to lineation on the foliation plane (Y direction), normal to foliation (Z direction)) define the reference frame.

Foliation is usually used to define a reference plane and *lineation* is used define a reference direction on the foliation plane. Foliation is a planar feature in a given rock, but its origin can be various (Hobbs *et al.*, 1976). The foliation plane may be defined by a plane normal to the maximum shortening strain (Fig. 1.7). Foliation can also be caused by compositional layering, grain-size variation and the orientation of platy minerals such as mica. When deformation is heterogeneous, such as the case for S-C mylonite (Lister and Snoke, 1984), one can identify two planar structures, one corresponds to the strain ellipsoid (a plane normal to maximum shortening, ε_3) and another to the shear plane.

Lineation is a linear feature that occurs repetitively in a rock. In most cases, the lineation is found on the foliation plane, although there are some exceptions. The most common is mineral lineation, which is defined by the alignment of non-spherical minerals such as clay minerals. The alignment of spinel grains in a spinel lherzolite and recrystallized orthopyroxene in a garnet lherzolite are often used to define the lineation in peridotites. One cause of lineation is strain, and in this case, the direction of lineation is parallel to the maximum elongation direction. However, there are a number of other possible causes for lineation including the preferential growth of minerals (e.g., Hobbs *et al.*, 1976).

Consequently, the interpretation of the significance of these reference frames (foliation/lineation) in natural rocks is not always unique. In particular, the question of growth origin versus deformation origin, and the strain ellipsoid versus the shear plane/shear direction can be elusive in some cases. Interpretation and identification of foliation/lineation become more difficult if the deformation geometry is not constant with time. Consequently, it is important to state clearly how one defines foliation/lineation in the structural analysis of a deformed rock. For more details on foliation and lineation, a reader is referred to a structural geology textbook such as Hobbs *et al.* (1976).

1.2.6. Various deformation geometries

The geometry of strain is completely characterized by the principal strain, and therefore a diagram such as the Flinn diagram (Fig. 1.6) can be used to define strain. However, in order to characterize the geometry of deformation completely, it is necessary to characterize the deformation gradient tensor (d_{ij} $(=\varepsilon_{ij}+\omega_{ij})$). Therefore the rotational component (vorticity tensor), ω_{ij}, must also be characterized. In this connection, it is important to distinguish between *irrotational* and *rotational deformation geometry*. Rotational deformation geometry refers to deformation in which $\omega_{ij}\neq 0$, and irrotational deformation geometry corresponds to $\omega_{ij}=0$. The distinction between them is important at finite strain. To illustrate this point, let us consider two-dimensional deformation (Fig. 1.8). For irrotational deformation, the orientations of the principal axes of strain are always parallel to those of principal stress. Therefore such a deformation is called *coaxial deformation*. In contrast, when deformation is rotational, such as *simple shear*, the orientations of principal axes of strain rotate progressively with respect to those of the stress (see Problem 1.5). This type of