# 1

# The Black–Scholes Theory of Derivative Pricing

The aim of this first chapter is to review the basic objects, ideas, and results of the classical Black–Scholes theory of derivative pricing. It is intended for readers who want to enter the subject or simply refresh their memory. This is not a complete treatment of this theory with detailed proofs but rather an intuitive but precise presentation including a few key calculations. Detailed presentations of the subject can be found in many books at various levels of mathematical rigor and generality, a few of which we list in the notes at the end of the chapter.

This book is about extending the Black–Scholes theory using perturbation methods in order to handle markets with stochastic volatility. The notation and many of the tools used in the constant volatility case will be used for the more complex markets throughout the book.

## 1.1 Market Model

In this simple model, suggested by Samuelson and used by Black and Scholes, there are two assets. One is a riskless asset (bond) with price $\beta_t$ at time $t$ described by the ordinary differential equation

$$d\beta_t = r\beta_t\, dt, \tag{1.1}$$

where $r$, a non-negative constant, is the instantaneous interest rate for lending or borrowing money. Setting $\beta_0 = 1$, we have $\beta_t = e^{rt}$ for $t \geq 0$. The price $X_t$ of the other asset, the risky stock or stock index, evolves according to the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t\, dW_t, \tag{1.2}$$

where $\mu$ is a constant mean return rate, $\sigma > 0$ is a constant *volatility*, and $(W_t)_{t\geq 0}$ is a standard *Brownian motion*. This fundamental model and the intuitive content of equation (1.2) are presented in the following sections.

### *1.1.1 Brownian Motion*

Brownian motion is a stochastic process whose definition, existence, properties, and applications were the subject of numerous studies during the twentieth century (and still are, in the twenty-first). Our goal here is to give a very intuitive and practical presentation.

A Brownian motion is a real-valued stochastic process with continuous trajectories that have independent and stationary increments. The trajectories are denoted by $t \to W_t$ and for the standard Brownian motion, we have that:

- $W_0 = 0$;
- for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}})$ are independent;
- for any $0 \le s < t$, the increment $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $\mathbb{E}\{(W_t - W_s)^2\} = t - s$. In particular, $W_t$ is $\mathscr{N}(0,t)$-distributed.

Denote by $(\Omega, \mathscr{F}, \mathbb{P})$ the probability space where our Brownian motion is defined and the expectation $\mathbb{E}\{\cdot\}$ is computed. For example, it could be $\Omega = \mathscr{C}([0,+\infty) : \mathbb{R})$, the space of all continuous trajectories $\omega$, with $W_t(\omega) = \omega(t)$. The $\sigma$-algebra $\mathscr{F}$ contains all sets of the form $\{\omega \in \Omega : |\omega(s)| < R, s \le t\}$; the Wiener measure, $\mathbb{P}$, is the probability distribution of the standard Brownian motion.

The increasing family of $\sigma$-algebras $\mathscr{F}_t$ generated by $(W_s)_{s \le t}$, the information on $W$ up to time $t$, and all the sets of probability 0 in $\mathscr{F}$, is called the *natural filtration* of the Brownian motion. This *completion* by the null sets is important, in particular for the following reason. If two random variables $X$ and $Y$ are equal almost surely ($X = Y$ $\mathbb{P}$-a.s. means $\mathbb{P}\{X = Y\} = 1$) and if $X$ is $\mathscr{F}_t$-measurable (meaning that any event $\{X_t \le x\}$ belongs to $\mathscr{F}_t$) then $Y$ is also $\mathscr{F}_t$-measurable.

A stochastic process $(X_t)_{t \ge 0}$ is *adapted* to the filtration $(\mathscr{F}_t)_{t \ge 0}$ if the random variable $X_t$ is $\mathscr{F}_t$-measurable for every $t$. We say that $(X_t)$ is $(\mathscr{F}_t)$-adapted. If another process $(Y_t)$ is such that $X_t = Y_t$ $\mathbb{P}$-a.s. for every $t$ then it is also $(\mathscr{F}_t)$-adapted.

The independence of the increments of the Brownian motion and their normal distribution can be summarized using *conditional characteristic functions*. For $0 \le s < t$ and $u \in \mathbb{R}$

Cambridge University Press & Assessment
978-0-521-84358-4 — Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives
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$$\mathbb{E}\left\{e^{iu(W_t-W_s)} \mid \mathscr{F}_s\right\} = e^{-\frac{u^2(t-s)}{2}}. \tag{1.3}$$

If $W$ is a Brownian motion, by independence of the increment $W_t - W_s$ from the past $\mathscr{F}_s$, the left-hand side of (1.3) is simply $\mathbb{E}\left\{e^{iu(W_t-W_s)}\right\}$, which is the characteristic function of a centered normal random variable with variance $t-s$, and is equal to the right-hand side. Conversely, if (1.3) holds, then the continuous process $(W_t)$ is a standard Brownian motion.

This independence of increments makes the Brownian motion an ideal candidate for defining a complete family of independent infinitesimal increments $dW_t$, which are centered, normally distributed with variance $dt$ and which will serve as a model of (Gaussian white) noise. The drawback is that the trajectories of $(W_t)$ cannot be "nice" in the sense that they are not of bounded variation, as the following simple computation suggests. Let $t_0 = 0 < t_1 < \cdots < t_n = t$ be a subdivision of $[0,t]$, which we may suppose evenly spaced so that $t_i - t_{i-1} = t/n$ for each interval. The quantity

$$\mathbb{E}\left\{\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right\} = n\mathbb{E}\{|W_{\frac{t}{n}}|\} = n\sqrt{\frac{t}{n}}\mathbb{E}\{|W_1|\}$$

goes to $+\infty$ as $n \nearrow +\infty$, indicating that the integral with respect to $dW_t$ cannot be defined in the usual way "trajectory by trajectory." We describe how such integrals can be defined in the next section.

### *1.1.2 Stochastic Integrals*

For $T$ a fixed finite time, let $(X_t)_{0\le t\le T}$ be a continuous stochastic process adapted to $(\mathscr{F}_t)_{0\le t\le T}$, the filtration of the Brownian motion up to time $T$, such that

$$\mathbb{E}\left\{\int_0^T X_t^2 dt\right\} < +\infty. \tag{1.4}$$

Using iterated conditional expectations and the independent increments property of Brownian motion, we note that with $t_0 < t_1 < \cdots < t_n = t$

$$\mathbb{E}\left\{\left(\sum_{i=1}^{n} X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right)\right)^2\right\} = \mathbb{E}\left\{\sum_{i=1}^{n}\left(X_{t_{i-1}}\right)^2(t_i - t_{i-1})\right\},$$

for $t \le T$, which is a basic calculation in the construction of stochastic integrals. Note also that the Brownian increments on the left are forward in time and that the sum on the right converges to $\mathbb{E}\left\{\int_0^t X_s^2 ds\right\}$, which is finite by (1.4).

The *stochastic integral* of $(X_t)$ with respect to the Brownian motion $(W_t)$ is defined as a limit in the mean-square sense ($L^2(\Omega)$)

$$\int_0^t X_s dW_s = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} X_{t_{i-1}} \left( W_{t_i} - W_{t_{i-1}} \right), \tag{1.5}$$

as the mesh size of the subdivision goes to zero.

As a function of time $t$, this stochastic integral defines a continuous square integrable process such that

$$\mathbb{E}\left\{ \left( \int_0^t X_s dW_s \right)^2 \right\} = \mathbb{E}\left\{ \int_0^t X_s^2 ds \right\}, \tag{1.6}$$

and has the *martingale property*

$$\mathbb{E}\left\{ \int_0^t X_u dW_u \mid \mathscr{F}_s \right\} = \int_0^s X_u dW_u \quad \mathbb{P}\text{-a.s., for } s \le t, \tag{1.7}$$

as can easily be deduced from the definition (1.5). The *quadratic variation* $\langle Y \rangle_t$ of the stochastic integral $Y_t = \int_0^t X_u dW_u$ is

$$\langle Y \rangle_t = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} (Y_{t_i} - Y_{t_{i-1}})^2 = \int_0^t X_s^2 ds \tag{1.8}$$

in the mean-square sense.

Stochastic integrals are zero-mean, continuous, and square integrable martingales. It is interesting to note that the converse is also true: every zero-mean, continuous, and square integrable martingale is a Brownian stochastic integral. This representation result will be made precise and used in Section 1.4.

### *1.1.3 Risky Asset Price Model*

The Black–Scholes model for the risky asset price corresponds to a continuous process $(X_t)$ such that, in an infinitesimal amount of time $dt$, the infinitesimal return $dX_t/X_t$ has mean $\mu dt$, proportional to $dt$, with a constant *rate of return* $\mu$, and centered random fluctuations independent of the past up to time $t$. These fluctuations are modeled by $\sigma dW_t$, where $\sigma$ is a positive constant *volatility* which measures the strength of the noise, and $dW_t$ the infinitesimal increments of the Brownian motion. The corresponding formula for the infinitesimal return is

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t, \tag{1.9}$$

which is the stochastic differential equation (1.2). The right-hand side has the natural financial interpretation of a return term plus a risk term. We are also assuming that there are no dividends paid in the time interval that we are considering. It is easy to incorporate a continuous dividend rate in all that follows, but for simplicity we shall omit this here.

In integral form, this equation is

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s, \tag{1.10}$$

where the last integral is a stochastic integral as described in Section 1.1.2 and where $X_0$ is the initial value, which is assumed to be independent of the Brownian motion and square integrable.

Equation (1.10), or (1.2) in the differential form, is a particular case of a general class of stochastic differential equations driven by a Brownian motion:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{1.11}$$

or in integral form

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{1.12}$$

In the Black–Scholes model, $\mu(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$; these are independent of $t$, differentiable in $x$, and linearly growing at infinity (since they are linear). This is enough to ensure existence and uniqueness of a continuous adapted and square integrable solution $(X_t)$. The proof of this result is based on simple estimates like

$$\begin{aligned} \mathbb{E}\left\{X_t^2\right\} &= \mathbb{E}\left\{\left(X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s\right)^2\right\} \\ &\leq 3\left(\mathbb{E}\left\{X_0^2\right\} + (\mu^2 T + \sigma^2)\int_0^t \mathbb{E}\left\{X_s^2\right\} ds\right), \end{aligned}$$

where we used the inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, the Cauchy–Schwarz inequality

$$\mathbb{E}\left(\int_0^t X_s ds\right)^2 \leq t \int_0^t \mathbb{E}\left\{X_s^2\right\} ds,$$

and (1.6). We deduce

$$0 \leq \mathbb{E}\left\{X_t^2\right\} \leq c_1 + c_2 \int_0^t \mathbb{E}\left\{X_s^2\right\} ds,$$

for $0 \leq t \leq T$ and constants $c_1$ and $c_2 \geq 0$. By a direct application of Gronwall's lemma, we deduce that the solution is *a priori* square integrable. The construction of a solution and the proof of uniqueness can be

Cambridge University Press & Assessment
978-0-521-84358-4 — Multiscale Stochastic Volatility for Equity, Interest Rate, and Credit Derivatives
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obtained by similar and slightly more complicated estimates that use the Kolmogorov–Doob inequality for martingales.

Looking at equation (1.9), it is very tempting to write $X_t/X_0$ explicitly as the exponential of $(\mu t + \sigma W_t)$. However, this is not correct because the usual chain rule is not valid for stochastic differentials. For instance $W_t^2$ is not equal to $2\int_0^t W_s dW_s$ as might be expected since, by the martingale property (1.7), this last integral has an expectation equal to zero but $\mathbb{E}\left\{W_t^2\right\} = t$.

This discrepancy is corrected by Itô's formula, which we explain now.

### *1.1.4 Itô's Formula*

A function of the Brownian motion $W_t$ defines a new stochastic process $g(W_t)$. We suppose in the following that the function $g$ is twice continuously differentiable, bounded, and has bounded derivatives. The purpose of the chain rule is to compute the differential $dg(W_t)$, or equivalently its integral $g(W_t) - g(W_0)$. Using the subdivision $t_0 = 0 < t_1 < \cdots < t_n = t$ of $[0,t]$, we write

$$g(W_t) - g(W_0) = \sum_{i=1}^{n} (g(W_{t_i}) - g(W_{t_{i-1}})).$$

We then apply Taylor's formula to each term to obtain

$$\begin{aligned} g(W_t) - g(W_0) = &\sum_{i=1}^{n} g'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\ &+ \frac{1}{2}\sum_{i=1}^{n} g''(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 + R, \end{aligned}$$

where $R$ contains all the higher-order terms.

If $(W_t)$ were differentiable only the first sum would contribute to the limit as the mesh size of the subdivision goes to zero, leading to the chain rule $dg(W_t) = g'(W_t)W_t' dt$ of classical calculus. In the Brownian case, $(W_t)$ is not differentiable and, by (1.5), the first sum converges to the stochastic integral

$$\int_0^t g'(W_s) dW_s.$$

The correction comes from the second sum which, like (1.8), converges to

$$\frac{1}{2}\int_0^t g''(W_s) ds,$$

as can be seen by comparing it in $L^2$ with $\frac{1}{2}\sum_{i=1}^{n} g''(W_{t_{i-1}})(t_i - t_{i-1})$. The higher-order terms contained in $R$ converge to zero and do not contribute to the limit, which is

$$g(W_t) - g(W_0) = \int_0^t g'(W_s)dW_s + \frac{1}{2}\int_0^t g''(W_s)ds. \tag{1.13}$$

This is the simplest version of Itô's formula. It is often written in differential form:

$$dg(W_t) = g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt. \tag{1.14}$$

The next step is deriving a similar formula for $dg(X_t)$, where $X_t$ is the solution of a stochastic differential equation like (1.11). We give here this general formula for a function $g$ depending also on time $t$:

$$dg(t,X_t) = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)d\langle X\rangle_t, \tag{1.15}$$

where $dX_t$ is given by the stochastic differential equation (1.11) and

$$\langle X\rangle_t = \int_0^t \sigma^2(s,X_s)ds$$

is the quadratic variation of the martingale part of $X_t$: that is, of the stochastic integral on the right-hand side of (1.12). In terms of $dt$ and $dW_t$ the formula is

$$dg(t,X_t) = \left(\frac{\partial g}{\partial t} + \mu(t,X_t)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2 g}{\partial x^2}\right)dt + \sigma(t,X_t)\frac{\partial g}{\partial x}dW_t, \tag{1.16}$$

where all the partial derivatives of $g$ are evaluated at $(t,X_t)$.

As an application we can compute the differential of the discounted price $g(t,X_t) = e^{-rt}X_t$:

$$\begin{aligned} d\left(e^{-rt}X_t\right) &= -re^{-rt}X_t dt + e^{-rt}dX_t \\ &= e^{-rt}\left(-rX_t + \mu(t,X_t)\right)dt + e^{-rt}\sigma(t,X_t)dW_t, \end{aligned} \tag{1.17}$$

since the second derivative of $g(t,x) = xe^{-rt}$ with respect to $x$ is zero. In the particular case of the price $X_t$ given by (1.2), $\mu(t,x) = \mu x$ and $\sigma(t,x) = \sigma x$ so we obtain

$$d\left(e^{-rt}X_t\right) = (\mu - r)\left(e^{-rt}X_t\right)dt + \sigma\left(e^{-rt}X_t\right)dW_t. \tag{1.18}$$

The discounted price $\widetilde{X}_t = e^{-rt}X_t$ satisfies the same equation as $X_t$ where the return $\mu$ has been replaced by $\mu - r$.

### *1.1.5 Lognormal Risky Asset Price*

Coming back to the stochastic differential equation (1.9) for the evolution of the stock price $X_t$, it is natural to suspect from the ordinary calculus formula $\int dx/x = \log x$ that $\log X_t$ might satisfy an equation that we can integrate explicitly. We compute the differential of $\log X_t$ by applying Itô's formula (1.16) with $g(t,x) = \log x$, $\mu(t,x) = \mu x$, and $\sigma(t,x) = \sigma x$:

$$d \log X_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t.$$

The logarithm of the stock price is then given explicitly by

$$\log X_t = \log X_0 + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t,$$

which leads to the following formula for the stock price:

$$X_t = X_0 \exp\left( (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right). \tag{1.19}$$

The return $X_t/X_0$ is *lognormal*: it is the exponential of a nonstandard Brownian motion which is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right) t$ and variance $\sigma^2 t$ at time $t$. The process $(X_t)$ is also called *geometric* Brownian motion. The stock price given by (1.19) satisfies equation (1.9). It can also be obtained as a diffusion limit of binomial tree models which arise when Brownian motion is approximated by a random walk.

Notice that, if $X_0 = 0$, $X_t$ stays at zero at all times thereafter. Thus in this model, bankruptcy (zero stock price) is a permanent state. However, $W_t$ is finite at all times, and therefore, if $X_0 > 0$, $X_t$ remains positive at all times.

In Figure 1.1, we show a sample path or realization of a geometric Brownian motion $(X_t)$ in which $\mu = 0.15$, $\sigma = 0.1$, and $X_0 = 95$. This path exhibits the "average growth plus noise" behavior we expect from this model of asset prices.

### *1.1.6 Ornstein–Uhlenbeck Process*

Many financial quantities, volatility amongst them, are modeled as *mean-reverting* processes, a term we shall explain in more detail in Chapters 2 and 3. The simplest example of a mean-reverting diffusion is the Ornstein–Uhlenbeck process, defined as a solution of

$$dY_t = \alpha(m - Y_t)dt + \beta\, dW_t, \tag{1.20}$$

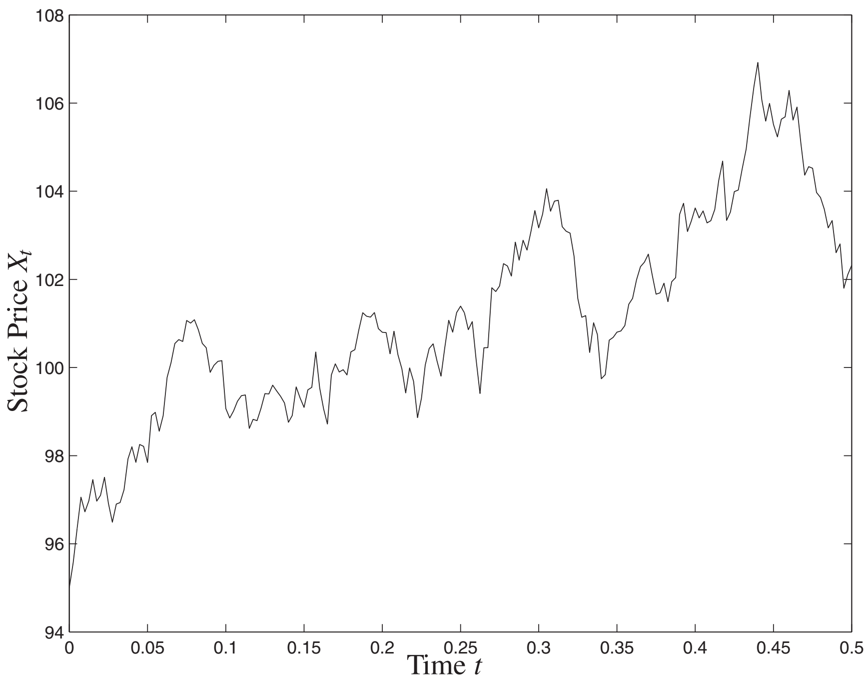


Figure 1.1 A sample path of a geometric Brownian motion defined by the stochastic differential equation (1.9), with $\mu = 0.15$, $\sigma = 0.1$, and $X_0 = 95$.

where $\alpha$ and $\beta$ are positive constants. This is one of the few explicitly solvable stochastic differential equations, which we illustrate here as an application of Itô's formula.

First, we rearrange the terms to write

$$dY_t + \alpha Y_t\,dt = \alpha m\,dt + \beta\,dW_t.$$

Multiplying through by the "integrating factor" $e^{\alpha t}$ gives

$$d(e^{\alpha t}Y_t) = \alpha m e^{\alpha t}\,dt + \beta e^{\alpha t}\,dW_t,$$

where the left-hand exact integral is easily checked from Itô's formula (1.15). Integrating from zero to $t$ and multiplying through by $e^{-\alpha t}$ gives

$$Y_t = m + (y-m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)}dW_s, \tag{1.21}$$

where $y$ is its (assumed known) starting value.

From this representation, it follows that $Y$ is a Gaussian process and the distribution of $Y_t$ is $\mathcal{N}\left(m + (y-m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})\right)$. Its long-run distribution, obtained as $t \to \infty$, is $\mathcal{N}(m, \beta^2/2\alpha)$, which does not depend on $y$.

The concept of long-run (invariant) distribution will be discussed in detail in Chapter 3.

## 1.2 Derivative Contracts

Derivatives are contracts based on the underlying asset price $(X_t)$. They are also called *contingent claims*. We will be interested primarily in *options*, which can be European, American, path-independent, or path-dependent. The definition of the options discussed in this first chapter is given in the following sections.

### 1.2.1 European Call and Put Options

A *European call option* is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined *strike price* $K$ on the *maturity* date $T$. If $X_T$ is the price of the underlying asset at maturity time $T$, then the value of this contract at maturity, its *payoff*, is

$$h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & \text{if } X_T > K, \\ 0 & \text{if } X_T \le K, \end{cases} \tag{1.22}$$

since in the first case the holder will exercise the option and make a profit $X_T - K$ by buying the stock for $K$ and selling it immediately at the market price $X_T$. In the second case the option is not exercised, since the market price of the asset is less than the strike price.

Similarly, a *European put option* is a contract that gives its holder the right, but not the obligation, to sell a unit of the asset for a strike price $K$ at the maturity date $T$. Its payoff is

$$h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T & \text{if } X_T < K, \\ 0 & \text{if } X_T \ge K, \end{cases} \tag{1.23}$$

since in the first case buying the stock at the market price and exercising the put option yields a profit of $K - X_T$, and in the second case the option is simply not exercised.

More generally, we will consider European derivatives defined by their maturity time $T$ and their non-negative payoff function $h(x)$. This will be a contract which pays $h(X_T)$ at maturity time $T$ when the stock price is $X_T$. The standard European-style derivatives are *path-independent* because the payoff $h(X_T)$ is only a function of the value of the stock price at maturity time $T$.