The Calculus of Retirement Income

Financial Models for Pension Annuities and Life Insurance

This book introduces and develops—from a unique financial perspective—the basic actuarial models that underlie the pricing of life-contingent pension annuities and life insurance. The ideas and techniques are then applied to the real-world problem of generating sustainable retirement income toward the end of the human life cycle. The roles of lifetime income, longevity insurance, and systematic withdrawal plans are investigated within a parsimonious framework. The underlying technology and terminology of the book are based on continuous-time financial economics, merging analytic laws of mortality with the dynamics of equity markets and interest rates. Nonetheless, the text requires only a minimal background in mathematics, and it emphasizes examples and applications rather than theorems and proofs. The Calculus of Retirement Income is an ideal textbook for an applied course on wealth management and retirement planning, and it can serve also as a reference for quantitatively inclined financial planners. This book is accompanied by material on the Web site (www.ifid.ca/CRI).

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The Calculus of Retirement Income

Financial Models for Pension Annuities and Life Insurance

MOSHE A. MILEVSKY

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