PART ONE

# MODELS OF ACTUARIAL FINANCE

ONE

# Introduction and Motivation

## 1.1 The Drunk Gambler Problem

A few years ago I was asked to give a keynote lecture on the subject of retirement income planning to a group of financial advisors at an investment conference that was taking place in Las Vegas. I arrived at the conference venue early—as most neurotic speakers do—and while I was waiting to go on stage, I decided to wander around the nearby casino, taking in the sights, sounds, and smells of flashy cocktail waitresses, clanging coins, and musty cigars. Although I'm not a fan of gambling myself, I always enjoy watching others get excited about the mirage of a hot streak before eventually losing.

On this particular random walk around the roulette tables, I came across a rather eccentric-looking player smoking a particularly noxious cigar, though seemingly aloof and detached from the action around him. As I approached that particular table, I noticed two odd things about *Jorge*; a nickname I gave him. First, Jorge appeared to be using a very primitive gambling strategy. He was sitting in front of a large stack of red $5 chips, and on each spin of the wheel he would place one—and only one—of those $5 chips as a bet on the black portion of the table. For those of you who aren't familiar with roulette, this particular bet would double his money if the spinning ball landed on any one of the 18 black numbers, but it would cost him his bet if the ball came to a halt on any of the 18 red numbers or the occasional 2 green numbers. This is the simplest of all possible bets in the often complicated world of casino gambling: black, you win; red or green, you lose.

Yet, watching him closely over a number of spins, I noticed that—regardless of whether the ball landed on a black number (yielding a $10 payoff for his $5 gamble) or landed on red or green numbers (causing a loss of his original $5 chip)—he would continue mechanically to bet a $5 chip on black for each consecutive round. This seemed rather boring and pointless to me. Most gamblers double up, get cautious, react to past outcomes, and take advantage of what they suppose is a hot streak. Rarely do they do the exact same thing over and over again.

Even more peculiar to me was what Jorge was doing in between roulette rounds, while the croupier was settling the score with other players and getting ready for the next spin. In one swift motion, Jorge would lift a rather large drinking glass filled with some unknown (presumably alcoholic) beverage, take a deep gulp, and then put the glass back down next to him. But, immediately upon his glass touching the green velvet surface, a waitress would top up the drinking glass and Jorge would mechanically hand her one of the $5 chips from his stack of capital. *This process continued after each and every spin of the wheel*. Try to imagine this for a moment. The waitress waits around for the wheel to stop spinning so that she can pour Jorge another round of gin—or perhaps it was scotch—so that she can get yet another $5 tip from this rather odd-looking character.

As I was standing there mesmerized by Jorge's hypnotic actions and repeated drinking, I couldn't help but wonder whether Jorge would pass out drunk and fall off his stool before he could cash in what was left of his chips.

There was no doubt in my mind that, if he continued with the same strategy, his stash of casino chips would continue to dwindle and eventually disappear. Note that after each round of spinning and drinking, his investment capital would either remain unchanged or would decline by $10. If the ball landed on black and he then paid $5 for the drink, he would be back where he started. If the ball landed on red and he then paid $5, the total loss for that round was $10. Thus, his pile of chips would never grow. The pattern went something like this: 26 chips, 26 chips, 26 chips, 24 chips, 22 chips, 22 chips, 20 chips, and so forth.

In fact, I was able to develop a simple model for calculating the odds that Jorge would run out of chips before he ran out of sobriety. From where I was standing, it appeared that he had about 20 more chips or $100 worth of cash. There was a 47.4% chance (18/38) he would get lucky with black on any given spin, and I loosely assumed a 10% chance he would pass out with any swig from the glass. Working out the math—and I promise to do this in detail in Section 1.6—there is a 15% chance he'd go bankrupt while he was still sober. Stated from the other side, I estimated an 85% chance he would pass out and fall off his chair before his stack of chips disappeared. That would be interesting to observe. Obviously, the model is crude and the numbers are rounded—and perhaps Jorge could hold his liquor better than I assumed—but I can assure you the waitress wanted Jorge's blacks to last forever.

I was planning to stick around to see whether my statistical predictions would come true, but time was running short and I had to return to my speaking engagement. As I was rushing back, weaving through the many tables, it occurred to me that I had just experienced a quaint metaphor on financial planning and risk management as retirees approach the end of the human life cycle.

With just a bit of imagination, think of what happens to most people as they reach retirement after many years of work—and hopefully with a bit of savings—but with little prospect for future employment income. They start retirement with a stack of chips that are invested (wagered or allocated) among various asset classes such as stocks, bonds, and cash. Each week, month, or year the retirees must withdraw or redeem some of those chips in order to finance their retirement income. And, whether the roulette wheel has landed on black (a bull market) or on green or red (flat or bear market), a retiree must consume. If the retiree lives for a very long time, there is a much greater chance that the chips will run out. If, on the other hand, the retiree spends only five or ten years at the retirement table, the odds are that the money will last. The retiree can obviously control the number of chips to be removed from the table (i.e., the magnitude of retirement income) as well as the riskiness of the bets (i.e., the amount allocated to the various investments). Either way, it should be relatively easy to compute the probability that a given investment strategy and a given consumption strategy will lead to retirement ruin.

So, in some odd way, we are all destined to be Jorge.

## 1.2 The Demographic Picture

In mid-2005 there are approximately 36 million Americans above the age of 65, which is approximately 13% of the population. By the year 2030 this number is expected to double to 70 million. Indeed, the fastest-growing segment of the elderly population is the group of those 85+ years old. The aging of the population is a global phenomenon, and many from the over-65 age group will continue working on a part-time basis well into their late sixties and seventies. A fortunate few will have earned a defined benefit (DB) pension that provides income for the rest of their natural life. Most others will have likely participated in a defined contribution (DC) plan, which places the burden of creating a pension (annuity) on the retiree. All of these retirees will have to generate a retirement income from their savings and their pension wealth. How they should do this at a sustainable rate—and what they should do with the remaining corpus of funds—is the impetus for this book.

Table 1.1 provides some hard evidence, as well as some projections, on the potential size and magnitude of the retirement income "problem." Using

**Table 1.1.** *Old-age dependency ratio[a] around the world*

| Country | Year[b] 2000 | 2010 | 2030 |
|---|---|---|---|
| Australia | 29.1% | 34.7% | 51.4% |
| Austria | 36.6% | 42.9% | 77.3% |
| Belgium | 40.5% | 44.7% | 68.5% |
| Canada | 29.1% | 35.2% | 58.8% |
| Denmark | 35.3% | 45.5% | 65.0% |
| Finland | 35.9% | 47.0% | 70.6% |
| France | 37.9% | 43.0% | 63.0% |
| Germany | 41.8% | 46.0% | 76.5% |
| Greece | 42.5% | 46.8% | 69.2% |
| Ireland | 28.0% | 30.7% | 42.5% |
| Italy | 42.7% | 49.7% | 78.5% |
| Japan | 41.4% | 58.4% | 79.0% |
| Mexico | 13.9% | 16.2% | 28.7% |
| New Zealand | 28.6% | 33.9% | 54.9% |
| Poland | 29.8% | 31.4% | 50.8% |
| South Korea | 18.3% | 23.9% | 53.0% |
| Spain | 38.2% | 42.2% | 69.7% |
| Sweden | 41.7% | 51.0% | 72.5% |
| Switzerland | 37.6% | 48.9% | 84.4% |
| Turkey | 16.4% | 17.8% | 28.6% |
| United Kingdom | 38.1% | 43.3% | 66.1% |
| United States | 29.3% | 33.2% | 52.0% |

[a] Size of population aged at least 60 divided by size of population aged 20–59.
[b] Figures for 2010 and 2030 are estimated.
*Source:* United Nations.

data compiled by the United Nations across different countries, the table shows the number of people above age 60 as a fraction of the (working) population between the ages of 20 and 59. The larger the ratio, the greater the proportion of retirees in a given country. This ratio is often called the old-age dependency ratio, since traditionally the older people within a society are dependent on the younger (working) ones for financial and economic support. Stated differently, a larger dependency ratio creates a larger burden for the younger generation.

In the year 2000, the old-age dependency ratio hovered around 30% for the United States and Canada, but by 2030 this number will jump to 52% in the United States and to 59% in Canada, according to UN estimates. At

**Table 1.2.** *Expected number of years spent in retirement around the world*

| Country | Males 2000 | Males 2010 | Males 2030 | Females 2000 | Females 2010 | Females 2030 |
|---|---|---|---|---|---|---|
| Australia | 19.0 | 19.7 | 21.0 | 27.1 | 27.8 | 29.1 |
| Austria | 21.1 | 22.1 | 23.8 | 27.3 | 28.6 | 30.2 |
| Belgium | 22.0 | 23.1 | 24.8 | 29.8 | 30.9 | 32.5 |
| Canada | 18.5 | 19.2 | 20.5 | 25.5 | 26.2 | 27.5 |
| Denmark | 17.3 | 18.0 | 19.3 | 22.9 | 24.1 | 25.7 |
| Finland | 20.3 | 20.9 | 22.3 | 25.2 | 26.0 | 27.2 |
| France | 20.5 | 21.4 | 23.2 | 26.7 | 27.5 | 29.0 |
| Germany | 19.4 | 20.2 | 22.1 | 25.3 | 26.6 | 28.2 |
| Greece | 18.4 | 18.9 | 20.2 | 23.7 | 24.4 | 25.7 |
| Ireland | 16.9 | 17.4 | 18.7 | 22.7 | 23.6 | 25.2 |
| Italy | 19.5 | 20.1 | 21.4 | 27.0 | 27.8 | 29.1 |
| Japan | 16.3 | 17.3 | 18.9 | 23.5 | 24.7 | 26.8 |
| New Zealand | 18.3 | 18.8 | 20.2 | 24.8 | 25.5 | 26.9 |
| Spain | 18.8 | 19.3 | 20.7 | 25.7 | 26.4 | 27.7 |
| Sweden | 18.7 | 19.4 | 20.6 | 23.2 | 23.9 | 25.4 |
| Switzerland | 16.6 | 17.2 | 18.4 | 24.3 | 24.9 | 26.2 |
| Turkey | 14.8 | 15.4 | 16.7 | 15.3 | 15.9 | 17.0 |
| United Kingdom | 18.0 | 18.9 | 20.5 | 23.8 | 25.0 | 26.8 |
| United States | 16.8 | 17.6 | 19.4 | 22.0 | 23.2 | 24.9 |

*Notes:* The actual retirement age varies by country. Figures for 2010 and 2030 are estimated.
*Sources:* Watson Wyatt and World Economic Forum.

the other extreme are countries like Mexico and Turkey, whose dependency ratios are currently in the low to mid-teens and should grow only to 28% by 2030. Despite the variations, these numbers are increasing in all countries.

According to a recent report prepared by the consulting firm of Watson Wyatt for the World Economic Forum, the main causes for the projected increases in the dependency ratio are a lower fertility ratio and the unprecedented increases in the length of human life. People live longer—beyond ages 60, 70, and 80, as demonstrated in Table 1.2—but they aren't born any earlier. So, the ratio of older people to younger people within any country continues to increase.

Human longevity is a fascinating topic in its own right. According to Dr. James Vaupel, Director of the Max Planck Institute for Demographic Research, the average amount of time that females live in the healthiest countries has been on the rise during the last 160 years at a steady pace of three months per year. For example, in 2005 Japanese women are estimated to have a life expectancy of approximately 85 years. Currently, Japanese women are the record holders when it comes to human longevity, and the projection is that—four years from now, in 2009—Japanese women will have a life expectancy of 86 years. Now let your imagination do the mathematics. What will the numbers look like in twenty or thirty years?

The oncoming wave of very long-lived retirees—who will possibly be spending more time in retirement than they did working—will require extensive and unique financial assistance in managing their financial affairs. Moreover, financial planners and investment advisors, who are on the front line against this oncoming wave, are hardly ignorant of this trend. Some have begun to retool themselves to better understand and meet the needs of this unique group of retirees. They are pressuring insurance companies, investment banks, and money managers to design, sell, and promote retirement income (a.k.a. pension) products that go beyond traditional assets.

For thirty years the financial services industry has focused on the *accumulation* phase for millions of active workers. Mutual fund and investment companies were falling all over themselves to provide guidance on the right mix of mutual funds, the right savings rate, and the most prudent level of risk to build the largest nest egg with the least amount of risk. The terms "asset allocation" and "savings rate" have become ubiquitous. Most investors understand the need for diversified investment portfolios.

What consumers and their advisors have less of an appreciation for are the interactions between longevity, spending, income, and the right investment portfolio. In part, the fault for this intellectual gap lies at the doorsteps of those instructors who teach portfolio theory within a static, one-period framework in which everybody lives to the end of the period. In fact, I have been teaching undergraduate, graduate, and doctoral students in business finance for over fifteen years and am continuously dismayed by their lack of knowledge about (and interest in) actuarial and insurance matters. Of course, learning about pensions or term life and disability insurance is not the most enjoyable activity when the competing course in the other lecture hall is teaching currency swap contracts, exotic derivatives, and hedge funds. Death and disability can't compete. For the most part, the students lack a framework that links the various ideas in a coherent manner. I hope this book helps make some of these actuarial issues more palatable and interesting to financial "quants."

Against this backdrop of financial demographics, product innovation, and human longevity, this book will attempt to merge the analytic language of modern financial theory with actuarial and insurance ideas motivated by what we may call the retirement income dilemma.

## 1.3 The Ideal Audience

The ideal audience for this book is … me. Yes, me. I know it might sound a bit odd, but writing this book has most basically given me a wonderful opportunity to collect and organize my thoughts on the topic of retirement income planning. I suspect that most authors will confirm a similar feeling and objective. Researching, organizing, and writing this book have helped me establish the financial and mathematical background needed to understand the topic with some rigor and depth. I am using this book also as a textbook for a graduate course I teach at the Schulich School of Business at York University (Toronto) on the topic of financial models for pension and insurance.

On a broader and more serious level, this book has two intended audiences. The first group consists of the growing legion of financial planners and investment advisors who possess a quantitative background or at least a numerical inclination. This group is in the daily business of giving practical advice to individual investors. They need a relevant and useful framework for explaining to their clients the risks they incur by either spending too much money in retirement, not having a diversified investment portfolio, or not hedging against the risks of underestimating their own longevity. And so I hope that the numerous stories, examples, tables, and case studies scattered throughout this book can provide an intuitive foundation for the underlying mathematical ideas. Yes, I know that some parts of the book, especially those involving calculus, may not be readily accessible to all. But as Dr. Roger Penrose—a world-renowned professor of mathematical physics at Oxford University—said in the introduction to his recent book *The Road to Reality: A Complete Guide to the Laws of the Universe*: "Do not be afraid to skip equations or parts of chapters when they begin to get a mite too turgid! I do this often myself …."

The second audience for this book consists of my traditional colleagues, peers, and fellow researchers in the area of financial economics, pensions, and insurance. There is a growing number of scholars around the world who are interested in furthering knowledge and practice by focusing on the *normative* aspects of finance for individuals. Collectively, they are creating scientific foundations for personal wealth management, quite similarly to the fine tradition of personal health management and the role of personal physicians. Indeed, work by such luminaries as Harry Markowitz (1991) and Robert Merton (2003) has emphasized the need for different tools when addressing personal financial problems as opposed to corporate financial problems.

## 1.4 Learning Objectives

This book is an attempt to provide a theory of applied financial planning over the human life cycle, with particular emphasis on retirement planning in a stochastic environment. My objective is not necessarily to analyze what people are doing or the positive aspects of whether they are rational, utility maximizing, and efficient in their decisions, but rather to provide the underlying analytic tools to help them and their advisors make better financial decisions. If I could sum up—in a half-joking manner—the educational objectives and underlying theme that run through this book, it would be to guide Jorge on his investment/gambling strategy so that he could continue tipping the waitress after every spin of the wheel and, it is hoped, pass out before his money is depleted. On a more serious note, this book is about developing the analytic framework and background models to help retiring individuals—and those who are planning for retirement—manage their financial affairs so that they can maintain a comfortable and dignified lifestyle during their golden years.

The main text consists of twelve chapters (an appendix of tables and a bibliography are also included). An ideal background for this book would be a basic understanding of the rules of differential and integral calculus, some basic probability theory, and familiarity with everyday financial instruments and markets.

Here is a brief chapter-by-chapter outline of what will be covered.

Part I Models of Actuarial Finance

1. *Introduction and Motivation*. This chapter.
2. *Modeling the Human Life Cycle*. I review the basic time value of money (TVM) mathematics in discrete time as it applies to the human life cycle. I present some deterministic models for computing the amount of savings needed during one's working years to fund a given standard of living during the retirement years. I briefly discuss how this relates to pension plans and the concept of retirement income replacement rates. The modeling is done without any need for calculus and requires only a basic understanding of algebra.

3. *Models of Human Mortality.* I introduce actuarial mortality tables and hazard rates using the tools of continuous-time calculus and probability. I present the analytic mortality workhorse of the book, which is the Gompertz–Makeham (GoMa) and exponential model for lifetime uncertainty. This chapter should help develop a thorough understanding of the *remaining lifetime* random variable, which is critical to all pension and insurance calculations.
4. *Valuation Models of Deterministic Interest.* I review the basics of continuous-time versus discrete interest rates as well as the term structure of interest rates. I provide valuation formulas for coupon bonds under a deterministic interest rate curve in continuous time. I introduce the concept of duration and convexity in continuous time and show how this can be used to approximate changes in bond prices.
5. *Models of Risky Financial Investments.* I develop models for understanding the long-term trade-off between risk and reward in the stock market. The analytics of portfolio diversification and the probability of losing money are examined. I start with some historical data and evidence on asset class investment returns. I then motivate portfolio growth rates and introduce the Brownian motion model underlying the lognormal distribution of investment returns. The chapter ends with a discussion of the difference between space and time diversification.
6. *Models of Pension Life Annuities.* I start by illustrating current market quotes of pension annuities and then move on to the valuation of life and pension annuities that provide income for the remainder of one's life. This is done by merging the concepts of interest rates, mortality rates, and pensions. This chapter can also be understood within the context of the valuation of bonds with a random maturity. The models are implemented for Gompertz–Makeham mortality; also, variable immediate annuities and joint life annuities are valued.
7. *Models of Life Insurance.* Features of real-world insurance prices and contracts are introduced. I then provide valuation formulas for basic term life insurance. I discuss how these formulas relate to pension annuities as well as the arbitrage relationship between them. Also discussed are the taxation treatment of insurance and its various permutations such as whole life, variable life, universal life, and so on.
8. *Models of DB vs. DC Pensions.* This chapter reviews the basic forms of public and private pensions. I develop some models for computing the value of a defined benefit (DB) pension promise and then compare this to a defined contribution (DC) pension. I discuss basic pension