Integer Partitions

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Chapter 1

Introduction

Mathematics as a human enterprise has evolved over a period of ten thousand years. Rock carvings suggest that the concepts of small counting numbers and addition were known to prehistoric cavemen. Later, the ancient Greeks invented such things as rational numbers, geometry, and the idea of mathematical proofs. Arab and Chinese mathematicians developed the handy positional system for writing numbers, as well as the foundation of algebra, counting with unknowns. From the Renaissance and onward, mathematics has evolved at an accelerating pace, including such immensely useful innovations as analytical geometry, differential calculus, logic, and set theory, until today's fruitful joint venture of mathematics and computers, each supporting the other.

We will delve into, or at least touch upon, many of these modern developments – but really, this book is about mathematical statements of a kind that would have made sense already to the cavemen! One could imagine a petroglyph or cave painting of the following kind:

```
O__                      O__ O__
 ||                       ||  ||
O__
 ||


O__                      O__O__O__
 ||                       || || ||
O__
 ||
O__
 ||
```

```
O__O__O__       O__ O__
 || || ||        ||  ||
                O__
                 ||


O__             O__O__O__O__
 ||              || || || ||
O__
 ||
O__
 ||
O__
 ||


O__O__O__       O__ O__O__
 || || ||        ||  || ||
O__             O__
 ||              ||
```

The concepts involved here are just small counting numbers, equality of numbers, addition of numbers, and the distinction between odd and even numbers. What is shown in the table is that for at least up to four animals, they can be lined up in rows of odd lengths in as many ways as in rows of different lengths. Written on today's blackboard instead of prehistoric rock, the table would have a more efficient design:

$1+1$	2
$1+1+1$ 3	3 $2+1$
$1+1+1+1$ $3+1$	4 $3+1$

The fact that there will always be as many items in the left column as in the right one was first proved by Leonhard Euler in 1748. But it is quite possible that someone observed the phenomenon earlier for small numbers, since it takes no more advanced mathematics than humans have accessed since the Stone Age. Nowadays, objects such as $3+1$ or $5+5+3+2$ are called *integer partitions*.

Stating it differently, an integer partition is a way of splitting a number into integer parts. By definition, the partition stays the same however we order the parts, so we may choose the convention of listing the parts from the largest part down to the smallest.

Euler's surprising result can now be given a more precise formulation: *Every number has as many integer partitions into odd parts as into distinct parts.* The table continues for five and six:

$1+1+1+1+1$ $3+1+1$ 5	5 $4+1$ $3+2$
$1+1+1+1+1+1$ $3+1+1+1$ $3+3$ $5+1$	6 $5+1$ $4+2$ $3+2+1$

EXERCISE

1. Continue the table from seven up to ten and check for yourself that Euler was correct! See if you can obtain some intuition for why the numbers of integer partitions of the two kinds are always equal. (Difficulty rating: 1)

Statements of the flavor "every number has as many integer partitions of this sort as of that sort" are called *partition identities*. The above partition identity of Euler was the first, but there are many, many more. It is an intriguing fact that there are so many different and unexpected partition identities. Here is another, very famous, example: *Every number has as many integer partitions into parts of size 1, 4, 6, 9, 11, 14, ... as into parts of difference at least two.*

The numbers $1, 4, 6, 9, 11, 14, \ldots$ are best described as having last digit $1, 4, 6$, or 9. Another way to put it is that when these numbers are divided by 5, the remainder is 1 or 4. Counting with remainders is called *modular arithmetic* and will appear several times in this book. In fact, it is striking that partition identities, their proofs and consequences, involve such a wide range of both elementary and advanced mathematics, and even modern physics. We hope that you will find integer partitions so compellingly attractive that they will lure you to learn more about these related areas too.

The last identity above was found independently by Leonard James Rogers in 1894 and Srinivasa Ramanujan in 1913. The tale of this identity is rich and has some deeply human aspects, one of which is that Rogers was a relatively unknown mathematican for a long time until the amazing prodigy Ramanujan rediscovered his results twenty years later, thereby securing eternal fame (at least among mathematicians) also for Rogers. The field of integer partitions comes with an unusually large supply of life stories and anecdotes that are romantic or astonishing, or simply funny. They are best presented, and best appreciated, in conjunction with the mathematics itself. Welcome to the wonderful world of integer partitions!