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Synopsis

Two underlying themes run through this book: first, defining and analyzing the subject of quantitative finance in the conceptual and mathematical framework of quantum theory, with special emphasis on its path-integral formulation, and, second, the introduction of the techniques and methodology of quantum field theory in the study of interest rates.

No attempt is made to apply quantum theory in re-working the fundamental principles of finance. Instead, the term 'quantum' refers to the abstract mathematical constructs of quantum theory that include probability theory, state space, operators, Hamiltonians, commutation equations, Lagrangians, path integrals, quantized fields, bosons, fermions and so on. All these theoretical structures find natural and useful applications in finance.

The path integral and Hamiltonian formulations of (random) quantum processes have been given special emphasis since they are equivalent to, as well as independent of, the formalism of stochastic calculus – which currently is one of the cornerstones of mathematical finance. The starting point for the application of path integrals and Hamiltonians in finance is in stock option pricing. Path integrals are subsequently applied to the modelling of linear and nonlinear theories of interest rates as a two-dimensional quantum field, something that is beyond the scope of stochastic calculus. Path integrals have the additional advantage of providing a framework for efficiently implementing the mathematical procedure of renormalization which is necessary in the study of nonlinear quantum field theories.

The term 'Quantum Finance' represents the synthesis of the concepts, methods and mathematics of quantum theory, with the field of theoretical and applied finance.

To ease the reader's transition to the mathematics of quantum theory, and of path integration in particular, the presentation of new material starts in a few cases with well-established models of finance. New ideas are introduced by first carrying out the relatively easier exercise of recasting well-known results in the formalism of quantum theory, and then going on to derive new results. One unexpected advantage of this approach is that theorists, working in the field of finance – presently focussed on notions drawn from stochastic calculus and partial differential equations – obtain a formalism that completely parallels and mirrors stochastic calculus, and prepares the ground for a (smooth) transition to the mathematics of quantum field theory.

All important equations are derived from first principles of finance and, as far as possible, a complete and self-contained mathematical treatment of the main results is given. A few of the exactly soluble models that appear in finance are closely studied since these serve as exemplars for demonstrating the general principles of quantum finance. In particular, the workings of the path-integral and Hamiltonian formulations are demonstrated by concretely working out, in complete mathematical detail, some of the more instructive models. The models themselves are interesting in their own right, thus providing a realistic context for developing the applications of path integrals to finance.

The book consists of the following three major components:[1]

Fundamental concepts of finance

The standard concepts of finance and equations of option theory are reviewed in this component.

Chapter 2 is an 'Introduction to finance' that is meant for readers who are unfamiliar with the essential ideas of finance. Fundamental concepts and terminology of finance, necessary for understanding the particular set of equations that arise in finance, are introduced. In particular, the concepts of risk and return, time value of money, arbitrage, hedging and, finally, Treasury Bonds and fixed-income securities are briefly discussed.

Chapter 3 on 'Derivative securities' introduces the concept of financial derivatives and discusses the pricing of derivatives. The classic analysis of Black and Scholes is discussed, the mathematics of stochastic calculus briefly reviewed and the connection of stochastic processes with the Langevin equation is elaborated. A derivation from first principles is given of the price of a stock option with stochastic volatility. The material covered in these two chapters is standard, and defines the framework and context for the next two chapters.

Systems with finite number of degrees of freedom

In this part Hamiltonians and path integrals are applied to the study of stock options and stochastic interest rates models. These models are characterized by having

[1] The path-integral formulation of problems in finance opens the way for applying powerful computational algorithms; these numerical algorithms are a specialized subject, and are not addressed except for a passing reference in Section 5.16.

finite number of **degrees of freedom**, which is defined to be the **number of independent random variables at each instant of time** t. Examples of such systems are a randomly evolving equity $S(t)$ or the spot interest rate $r(t)$, each of which have one degree of freedom. All quantities computed for quantum systems with a finite number of degrees of freedom are completely finite, and do not need the procedure of renormalization to have a well-defined value.

In Chapter 4 on 'Hamiltonians and stock options', the problem of the pricing of derivative securities is recast as a problem of quantum mechanics, and the Hamiltonians driving the prices of options are derived for both stock prices with constant and stochastic volatility. The martingale condition required for risk-neutral evolution is re-expressed in terms of the Hamiltonian. Potential terms in the Hamiltonian are shown to represent a class of path-dependent options. Barrier options are solved exactly using the appropriate Hamiltonian.

In Chapter 5 on 'Path integrals and stock options', the problem of option pricing is expressed as a Feynman path integral. The Hamiltonians derived in the previous chapter provide a link between the partial differential equations of option pricing and its path-integral realization. A few path integrals are explicitly evaluated to illustrate the mathematics of path integration. The case of stock price with stochastic volatility is solved exactly, as this is a nontrivial problem which is a good exemplar for illustrating the subtleties of path integration.

Certain exact simplifications emerge due to the path-integral representation of stochastic volatility and lead to an efficient Monte Carlo algorithm that is discussed to illustrate the numerical aspects of the path integral.

In Chapter 6 on 'Stochastic interest rates' Hamiltonians and path integrals', some of the important existing stochastic models for the spot and forward interest rates are reviewed. The Fokker–Planck Hamiltonian and path integral are obtained for the spot interest rate, and a path-integral solution of the Vasicek model is presented.

The Heath–Jarrow–Morton (HJM) model for the forward interest rates is recast as a problem of path integration, and well-known results of the HJM model are re-derived using the path integral.

Chapter 6 is a preparation for the main thrust of this book, namely the application of quantum field theory to the modelling of the interest rates.

Quantum field theory of interest rates models

Quantum field theory is a mathematical structure for studying systems that have infinitely many degrees of freedom; there are many new features that emerge for such systems that are beyond the formalism of stochastic calculus, the most important being the concept of renormalization for nonlinear field theories. All the chapters in this part treat the forward interest rates as a quantum field.

In Chapter 7 on ‘Quantum field theory of forward interest rates’, the formalism of path integration is applied to a randomly evolving curve: the forward interest rates are modelled as a randomly fluctuating curve that is naturally described by quantum field theory. Various linear (Gaussian) models are studied to illustrate the theoretical flexibility of the field theory approach. The concept of psychological future time is shown to provide a natural extension of (Gaussian) field theory models. The martingale condition is solved for Gaussian models, and a field theory derivation is given for the change of numeraire. Nonlinear field theories are shown to arise naturally in modelling positive-valued forward interest rates as well as forward rates with stochastic volatility.

In Chapter 8 on ‘Empirical forward interest rates and field theory models’, the empirical aspects of the forward rates are discussed in some detail, and it is shown how to calibrate and test field theory models using market data on Eurodollar futures. The most important result of this chapter is that a so-called ‘stiff’ Gaussian field theory model provides an almost exact fit for the market behaviour of the forward rates. The empirical study provides convincing evidence on the efficacy of the field theory in modelling the behaviour of the forward interest rates.

In Chapter 9 on ‘Field theory of Treasury Bonds’ derivatives and hedging’, the pricing of Treasury Bond futures, bond options and interest caps are studied. The hedging of Treasury Bonds in field theory models of interest rates is discussed, and is shown to be a generalization of the more standard approaches. Exact results for both instantaneous and finite time hedging are derived, and a semi-empirical analysis of the results is carried out.

In Chapter 10 on ‘Field theory Hamiltonian of the forward interest rates’ the state space and Hamiltonian is derived for linear and nonlinear theories. The Hamiltonian formulation yields an exact solution of the martingale condition for the nonlinear forward rates, as well as for forward rates with stochastic volatility. A Hamiltonian derivation is given of the change of numeraire for nonlinear theories, of bond option price, and of the pricing kernel for the forward interest rates.

All chapters focus on the conceptual and theoretical aspects of the quantum formalism as applied to finance, with material of a more mathematical nature being placed in the Appendices that follow each chapter. In a few instances where the reader might benefit from greater detail the derivations are included in the main text, but in a smaller font size. The Appendix at the end of the book contains mathematical results that are auxiliary to the material covered in the book. The reason for including the Appendices is to present a complete and comprehensive treatment of all the topics discussed, and a reader who intends to carry out some computations would find this material useful. In principle, the Appendices and the derivations in smaller type can be skipped without any loss of continuity.

Part I

Fundamental concepts of finance

2

Introduction to finance

The field of economics is primarily concerned with the various forms of productive activities required to sustain the material and spiritual life of society. **Real assets**, such as capital goods, management and labour force, and so on, are necessary for producing goods and services required for the survival and prosperity of society.

The term **capital** denotes the economic value of the real assets of a society. In most developed economies, economic assets have a monetized form, and capital can be given a monetary value or paper form, called the money form of capital.

Finance is a branch of economics that studies the money (paper) form of capital. Uncertainty and risk are of fundamental importance in finance [87].

The main focus in this book is on financial assets and financial instruments. **Financial assets**, in contrast to real assets, are pieces of paper that entitle its holder to a claim on a fraction of the real assets, and to the income (if any) that is generated by these real assets. For example, a person owning a stock of a company is entitled to yearly dividends (if any), and to a pro rata fraction of the assets if the company liquidates.

What distinguishes finance from other branches of economics is that it is primarily an empirical discipline due to the demands of the finance industry. Vast quantities of financial data are generated every day, in addition to mountains of accumulated historical data. Unlike other branches of economics, the empirical nature of finance makes it closer to the natural sciences, since the financial markets impose the need for practical and transparent quantitative models that can be calibrated and tested.

A financial asset is also called a **security**, and the specific form of a financial asset – be it a stock or a bond – is called a **financial instrument**. A financial asset is at the same time a **financial liability** for the issuing party, since its profit and assets are to be divided amongst all the stockholders. Stocks and bonds are in positive net supply. Derivatives in contrast are in zero net supply since the number of people holding the derivative exactly equals the number of people selling these derivatives – and hence derivatives amount to a zero-sum game. The payoff to the holder of a derivative instrument equals minus the payoff for the seller of the instrument.

An investor can invest in financial assets as well as in real assets, such as real estate, gold or some other commodity [54].

The following are the three primary forms of financial instruments.

- **Equity**, or common **stocks** and **shares** represent a share in the ownership of a company. The possession of a share does not guarantee a return, but only a pro rata fraction of the dividends, usually declared if the company is profitable. The value of a share may increase or decrease over time, depending on the performance of the company, and hence the owner of equity is exposed to the risks faced by the company. The holder of a stock has only a limited liability of losing the initial investment. Hence, the value of a stock is **never negative**, with its minimum value being zero. Equity is a form of asset since the holder of equity is a net owner of capital.
- **Fixed income securities**, also called bonds, are issued by corporations and governments, and promise either a single fixed payment or a stream of fixed payments. Bonds are **instruments of debt**, and the issuer of a bond in effect takes a loan from the buyer of the bond, with the repayment of the debt usually being scheduled over a fixed time interval, called the **maturity** of the bond. There is a great variety of bonds, depending on the different periods of maturity and provisions for the repayment stream. For example, the holder of a five-year coupon US Treasury Bond is promised a stream of interest payments every six months, with the principal being repaid at the end of five years, whereas a holder of a zero coupon US Treasury Bond receives a single cash flow on the maturity of the bond. The risk in the ownership of a fixed-income security is often considered to be less than the ownership of equity since – short of the issuer going bankrupt – the owner of a fixed-income security is guaranteed a return as long as the owner can hold the instrument till maturity. However, due to interest rate risk, credit risk and currency risk for the bonds that are issued in a foreign currency, a bond portfolio can lose as much value, or even more, than a portfolio of equities.
- **Derivative securities** are, as the term indicates, financial assets that are **derived** from other financial assets. The payoff of a derivative security can depend, for example, on the price of a stock or another derivative.

The three primary forms of financial instruments can be combined in an almost endless variety of ways, leading to more complex instruments. For example, a **preferred stock** combines features of equity and debt instruments by entitling the investor to a fraction of the issuer's equity, and at the same time – similar to bonds – to a stream of (fixed) payments.

Theoretical finance takes as its subject the money (paper) form of capital, and is primarily concerned with the problems of the time value of money, risk and return, and the valuation of securities and assets. The creation and arbitrage-free pricing of new financial instruments to suit the myriad needs of investors is of increasing importance. The design, risk-return analysis and hedging of these instruments are issues that are central to finance, and comprise the field of financial engineering.

2.1 Efficient market: random evolution of securities

A financial **market** is where the buyer and seller of a financial asset meet to enact the transaction of buying and selling. If one buys (or agrees to buy) a financial asset, one is said to have a **long position** or is said to be **going long**. On the other hand, if one is selling a financial asset, one is said to be **shorting** the asset, or, equivalently, have a **short position**. If one sells an asset without actually owning it, one is said to be engaged in **short selling**; the repurchase date for short selling is usually some time in the future.

There are various forms in which any market is organized, with the primary ones being the following. A **direct market** is based on a direct search of the buyer and seller, the **brokered market** is one in which the brokers – for fees – match the buyer with the seller, and, lastly, the **auction market** is one in which buyers and sellers interact simultaneously in a centralized market [100]. Financial assets and instruments are traded in specialized markets known as the **financial markets**, which will be discussed in the next section.

The concept of an **efficient market** is of great importance in the understanding of financial markets, and is tied to the concept of the 'fair price' of a security. One expects that for a market in equilibrium, the security will have its fair price, and that investors will consequently not trade in it any further. When in equilibrium, an efficient market is one in which the prices of financial instruments have only small and temporary deviations from their fair price.

Efficient market is closely related to the concept of market information. What differentiates the various players in the market is the amount of market information that is available to each of them. Market information in turn consists of three components, namely: (a) historical data of the prices and returns of financial assets, (b) public domain data regarding all aspects of the financial assets and (c) information known privately to a few market participants. Based on these three categories of information, the concept of 'weak', 'semi-strong' and 'strong' forms of market efficiency can, respectively, be defined [23].

Intuitively speaking, an efficient market in effect means most of the buyers and sellers in the market have equal wealth and information, with no collection of buyers or sellers having any (unfair) advantage over the others. A precise statement of the efficient market hypothesis is the following

> For a financial market that is in equilibrium, none of the players, given their endowment and information, want to trade any further. For efficient markets,

> prices reveal available market information. The inflow of new information comes in randomly – in bits and pieces – causing random responses from the market players, due to the incomplete nature of the incoming information, and results in random changes in the prices of the various financial instruments.

It is worth emphasizing that a far-reaching conclusion of the efficient market hypothesis is that, once the market is in equilibrium, **changes in the prices** of all securities, upto a drift, are **random** [23]. The reason being that in an efficient market the prices of financial instruments have already incorporated all the market information, and resulted in equilibrium prices; any departures of the prices from equilibrium should be uncertain and unpredictable, with changes being equally likely to be above and below the equilibrium price.

Hence changes in the prices of financial instruments should be represented by **random variables**. Suppose the value of an equity at time t is represented by $S(t)$; then the **change** in the value of an equity is random, that is, dS/dt is modelled as a random variable; this in turn implies the security $S(t)$ itself is also a random variable, with its initial (deterministic) condition specified at some time t_0. The extent to which a security $S(t)$ is random is specified by a quantity called the **volatility** of the security, and is usually denoted by σ_S, or simply by σ. The greater the volatility of a security, the greater are the random fluctuations in the price of the security. A volatility of $\sigma = 0$ consequently implies that the security has no uncertainty in its future evolution.

The risk that all investors face is a reflection of the **random evolution** of financial instruments, and is ultimately a reflection of the manner in which (financial) markets incorporate all the relevant features of the underlying real economy.

The efficient market hypothesis does not imply that new information or important events do not move the market; rather, the hypothesis implies that unexpected or unanticipated new information **disturbs** the equilibrium of the market prices of various securities, and systematically moves them to a **new** set of equilibrium prices. Once equilibrium is reached, ordinary information will be available to almost all participants and hence will lead to random changes in the revealed prices of the financial instruments.

Is the efficient market hypothesis empirically testable? As pointed out in [23], there are **two hypotheses** implicit in the existence of an efficient market, namely the hypothesis of efficiency together with the hypothesis that the market is in a particular equilibrium. It is only this **joint hypothesis** – namely of market efficiency and equilibrium – that can be empirically tested and which often leads to spirited academic debates regarding the efficiency of financial markets.

The concept of market equilibrium is similar to the idea of equilibrium for a thermodynamic system. The positions and velocities of individual particles, analogous to the prices of financial instruments, are random even though the system itself is in equilibrium. Furthermore, the efficiency of the market is analogous to the efficiency of a thermodynamic heat engine. No one expects an actual heat engine to have 100% efficiency, and an efficiency of say 60–70% is fairly common. Similarly, even if a financial market is not fully efficient, it is often still justified to apply mathematical models based on this concept.

2.2 Financial markets

The financial markets are organized to trade in various forms of financial instruments. The major segmentation of the financial markets is into the **capital markets** and the **money markets**. Capital markets are structured to trade in the primary forms of financial instruments, namely in instruments of equity, debt and derivatives. Indexes are a part of the capital markets and are equal to the weighted average of a basket of securities of a particular market; given their importance and depth, indexes are treated as entities distinct from the capital markets. Money markets, properly speaking, belong to the debt market, but since money market instruments trade in highly liquid and short-term debt, cash and cash equivalents, foreign currency transactions and so on, a separate market is set up for such transactions.

The following is a breakdown of the main forms of the financial markets:

1 Capital markets
- Equity market: common stocks; preferred stocks.
- Debt market: treasury (government) notes and bonds; corporate and municipal bonds; mortgage-backed securities (MBS)
- Derivative market: options; forwards and futures

2 Indexes
- Equity indexes: Dow Jones and Standard and Poor's Indexes; Nikkei index; DAX Index; STI Index etc.
- Debt indexes: bond market indicators

3 Money markets
- Cash time deposits
- Treasury bills
- Certificates of deposit
- Commercial paper
- Eurodollar deposits: refers to US$ deposits in non-US banks, or in overseas branches of US banks.