1

Stratified fluids and waves

1.1 Introduction

Outside of theoretical interest in their peculiar properties, one of the main motivations for understanding the dynamics of internal gravity waves is that they occur naturally in the atmosphere and oceans. In the atmosphere, through transporting horizontal momentum from the ground upwards, internal waves influence wind speeds and consequently the thermal structure of the atmosphere. By contrast, internal waves in the ocean are primarily important as they affect mixing through the transport of energy. Although internal waves do not play a dominant role in the evolution of weather and climate, their influence is non-negligible: numerical simulations that do not include the effects of internal waves predict wind speeds and temperature in the atmosphere incorrectly and they do not account for the observed levels of turbulent diffusivity in the oceans. At the mesoscale in the atmosphere internal wave breaking is a source of clear-air turbulence and in the ocean, internal solitary waves influence biological activity over continental shelves through mass transport and mixing.

This chapter begins with a brief introduction to stratified fluids and internal gravity waves and then gives an overview of the structure of the atmosphere and oceans with mention of internal gravity wave phenomena in these fluids. In the following sections, we derive the equations describing the motion and thermodynamics of fluids and then make approximations relevant to internal gravity wave dynamics. The derivations are sometimes heuristic, aiming to provide physical intuition rather than emphasizing rigour. Finally, we review the mathematics used to describe the structure and evolution of periodic waves, wavepackets and modes.

1.2 Stratified fluids

A stratified fluid tends to move in horizontal layers. ('*Strata*' from modern Latin means 'layers'.) Vertical motions are inhibited because the density and pressure change with height in such a way that it costs energy to move against or with the direction of gravity. Put another way, vertically displaced fluid in a stratified fluid feels a buoyancy restoring force acting in a direction opposite to the displacement. In this sense, the force acts like a spring, which feels a contracting force when stretched and an expanding force when compressed.

Sometimes a stratified fluid is said to be ‘stably stratified’ to distinguish it from an ‘unstably stratified fluid’, such as boiling water, in which buoyancy forces cause relatively light fluid to rise and relatively dense fluid to sink. At the conceptual boundary between these two classes of stratification is ‘uniform’, ‘homogeneous’ or ‘neutrally stratified’ fluid: vertically displaced fluid experiences no buoyancy forces. We are primarily interested in stably stratified fluids in this book because only these support internal gravity waves.

Both liquids and gases can be stratified. We will consider these separately below.

1.2.1 Stratified liquids and the ocean

In preparing a salad dressing, oil and vinegar form a stratified fluid with oil floating in a layer above the more dense vinegar. Many cocktails are also stratified fluids with dense sugary or creamy liqueurs resting at the bottom of the glass and lighter aperitifs layered on top, as shown in Figure 1.1.

As well as describing the sequential stacking of slabs of fluid, stratified fluids also include liquids whose density decreases continuously with height. For example,

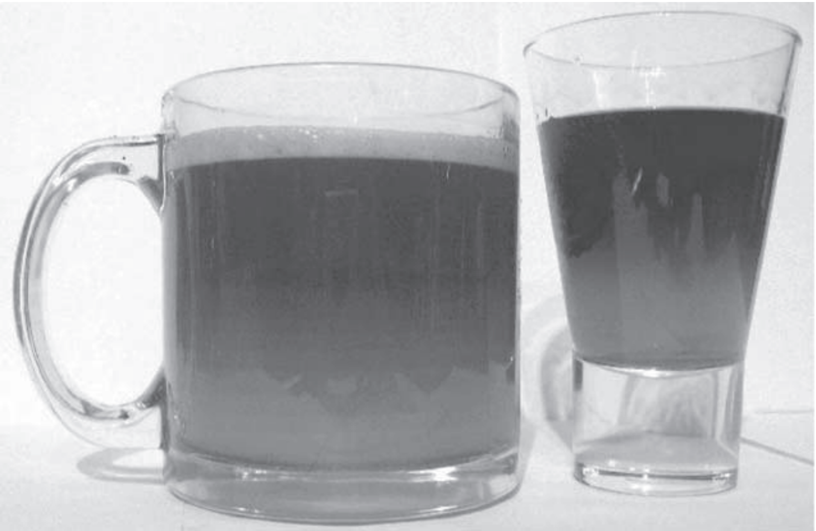

Fig. 1.1. Tasty stratified beverages.

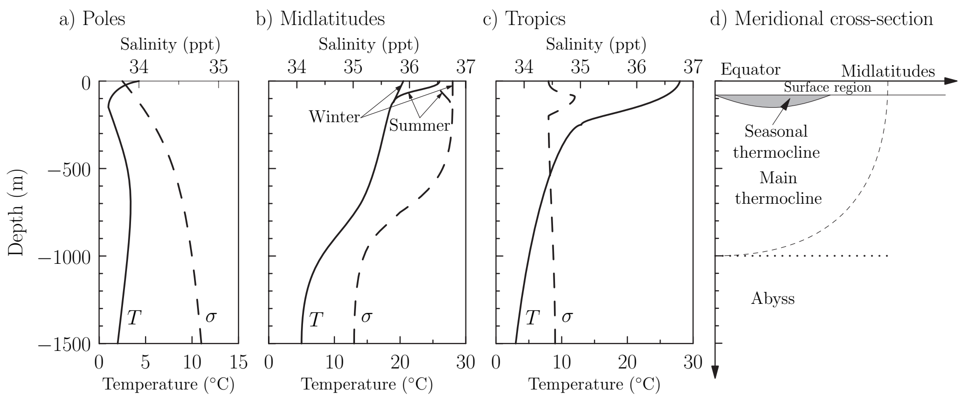

Fig. 1.2. Schematic showing vertical temperature (solid line) and salinity (dashed line) profiles of the ocean a) near the poles, b) at midlatitudes and c) near the equator. d) The vertical and meridional density structure of the ocean between the equator and poles showing the surface region, seasonal thermocline, main thermocline and the abyss. The seasonal thermocline is most pronounced in summer. In all plots the vertical axis is depth, as indicated in a).

saline solutions are ‘continuously stratified’ if the fluid becomes gradually less salty from bottom to top. Likewise, water is stratified by temperature if it becomes cooler with depth. The fluid is said to be ‘strongly stratified’ if the density decreases relatively rapidly with height, whereas it is ‘weakly stratified’ if the density decreases slowly with height.

The vertical structure of the ocean is effectively subdivided into layers according to the strength of its stratification. These are the surface mixed regions, the strongly stratified thermocline and the underlying weakly stratified abyss. These are illustrated schematically in Figure 1.2. Temperature, salinity and pressure change with depth but, except near estuaries, it is the temperature variation that is most important in determining the density change. In freshwater lakes, temperature alone determines the stratification.

Wind stress, surface wave breaking and convection lead to active turbulence near the ocean surface. This surface mixed region is unstratified or weakly stratified and generally extends downwards as far as 100 m depending upon latitude and season. The top 10 m of the ocean, which is most turbulent as a result of wind stress and surface wave breaking, is the surface skin region.

The thermocline region includes the seasonal thermocline and below it the main thermocline. Both extend from midlatitudes in the southern hemisphere to midlatitudes in the northern hemisphere. The seasonal thermocline is situated between approximately 50 and 150 m depth, depending upon season. Its stratification is strongest and its depth is shallowest in summer when vertical mixing is strongly inhibited by the large density contrast between the warm surface and cooler deep waters. In winter, surface cooling drives convection which deepens the thermocline, weakening its stratification and potentially, in lakes, eliminating it altogether. The main thermocline in the ocean, extending to approximately 1000 m depth, is unaffected by the seasons.

Strictly speaking a 'thermocline' is a highly stratified layer in which the temperature decreases rapidly with depth; this definition ignores salinity changes. A stratified layer resulting strictly from increasing saltiness with depth is called a 'halocline'.

The abyss lies between approximately 1000 m depth and the ocean floor, about 5000 m below the surface. This part of the ocean is weakly stratified and nearly quiescent, except where tidal flows move over the oceanic ridges and continental margins.

1.2.2 Stratified gases and the atmosphere

Gases as well as liquids form stratified fluids and share many of the properties of stratified liquids: just as warm water floats on cold, so does warm air hover near the ceiling and cold air near the floor of a room. When cold air formed by dry-ice creates clouds that pour off a stage it exposes a stratified flow of cold moisture-laden air in a relatively warm ambient.

On larger vertical scales the temperature of air can decrease with height and still be stably stratified. Thus cold air at a mountain top does not necessarily rush downslope into a warm valley. If it is pushed downward, it is compressed by higher pressures acting to increase its density and temperature. The compressed air continues downslope only if its increased temperature is nonetheless colder than its surroundings. If the temperature of the air is constant with height, vertically displaced air always feels a force that tends to restore it to its original position – it is stably stratified.

Generally, it is insufficient to say that a gas is stably stratified if its density decreases with height. Its stability depends primarily upon ambient pressure and temperature variations, as discussed in Section 1.7.1.

Like the ocean, the atmosphere is subdivided into layers depending upon the relative strength of the stratification at different altitudes. Immediately above the Earth's surface is the troposphere. Above that the stratosphere and mesosphere comprise what is called the middle atmosphere. Above these is the upper atmosphere composed of the thermosphere, which contains the ionosphere near its base. These are illustrated in Figure 1.3. Although the air masses evolve smoothly from one layer to the next, for conceptual convenience we separate the layers by boundaries.

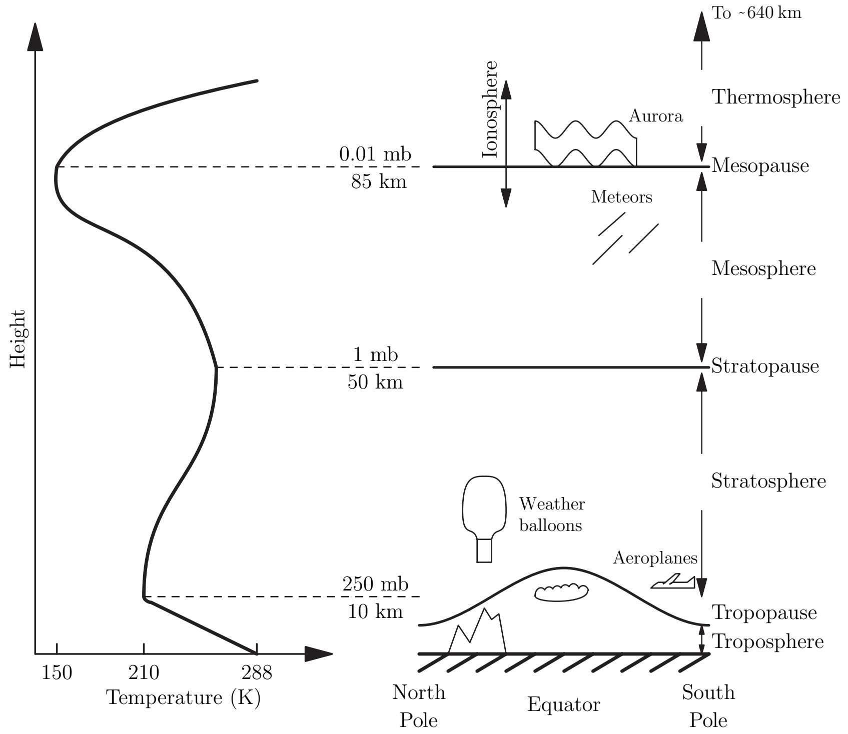

Fig. 1.3. Schematic representation of the main four layers of the atmosphere, which are determined by thermal variations with height.

Thus the top of the troposphere is the tropopause, the top of the stratosphere is the stratopause and the top of the mesosphere is the mesopause.

The troposphere, where our weather occurs, is neutrally stable or weakly stratified for the most part. Its relative homogeneity results from convection: during the day the sun warms the ground and the air above it mixes to about 7 km altitude near the poles and approximately 16 km near the equator.

In a neutrally stratified atmosphere, the temperature decreases with height at a rate of about 10°C per kilometre. This is known as the 'dry adiabatic lapse rate'. It means that if air containing no water vapour is lifted upwards by 1 km then its temperature will drop by 10°C as a result of cooling when it expands in the lower pressure surroundings.

The stratosphere extends from the tropopause to about 50 km altitude. It gets its name because it is relatively strongly stratified. One reason why commercial aeroplanes fly in the lower stratosphere is because vertical updrafts and turbulence are inhibited by the stable stratification. The stratification is strong due to chemistry: ozone, which is produced in large quantities in the equatorial stratosphere, efficiently absorbs solar radiation, heating the air so that its temperature increases with height.

The relatively weakly stratified mesosphere extends from the stratopause to between 80 and 85 km altitude. Ozone is less prominent in the mesosphere and so internal heating is weaker. Indeed, the coldest temperatures in the atmosphere are found in the mesosphere.

The thermosphere extends above the mesosphere to about 600 km altitude, beyond which is the exosphere and space. The internationally accepted boundary to outer space, the Kármán line, is in the lower thermosphere at 100 km. This is the approximate height at which air is too thin to provide lift to aircraft travelling slower than escape velocity. Trace amounts of oxygen and other ionized gases absorb intense solar radiation in the thermosphere and can raise temperatures to as high as 2000°C.

The ionosphere represents a region in the lower part of the thermosphere (and to a lesser extent in the mesosphere) where there is a relatively large concentration of charged particles. These are visible to the naked eye at night when there is strong solar activity forming aurora such as the Northern Lights.

1.3 Internal gravity waves

Surface waves on the ocean move up and down due to buoyancy. Water lifted upwards is heavier than its surroundings and falls downwards. It then overshoots its equilibrium position and experiences a restoring force that lifts the downward-displaced surface upwards again. This spring-like motion is felt collectively by the fluid which oscillates in space as well as time. Because buoyancy forces, and ultimately gravity, are necessary for the existence of surface waves, they are sometimes called gravity waves.

Likewise, the interface between warm and cold fluid or between fresh and salt water can oscillate forming what is called an ‘internal gravity wave’: a gravity wave that moves *within* a fluid. In a continuously stratified fluid, internal gravity waves are not confined to an interface and can in fact propagate vertically as well as horizontally through the fluid. Internal gravity waves at an interface and in continuously stratified fluid are shown in Figure 1.4.

Whether at an interface or moving within a continuously stratified fluid, internal gravity waves are sometimes more laconically called ‘gravity waves’ or ‘internal waves’. The former terminology is avoided since it is often specifically applied to surface waves. The latter terminology is used most often in this book to describe waves moving due to buoyancy in continuously stratified fluid. We use the term ‘interfacial waves’ to describe waves at the interface between hot and cold or fresh

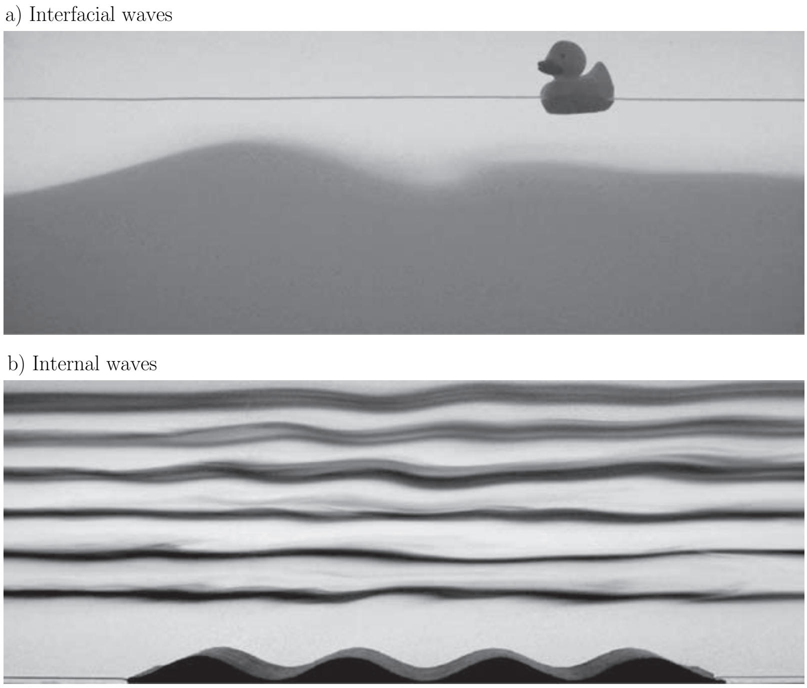

Fig. 1.4. a) An interfacial wave between fresh and underlying (dyed) salt water moving beneath a nearly flat surface, and b) internal waves in uniformly stratified fluid visualized by the displacement of dye lines spaced apart by 5 cm. The waves in b) result from the rightward translation of a model set of hills at the bottom of the image.

and salty water. Waves at a water–air interface are specifically referred to as 'surface waves' or 'water waves'.

In addition to buoyancy forces, internal gravity waves with periods comparable to a day feel the effects of the Earth's rotation through Coriolis forces. This class of waves we call 'inertial waves' if they occur at an interface and 'inertia gravity waves' if they occur in continuously stratified fluid. The terminology arises not because the waves possess inertia but because Coriolis forces arise from apparent accelerations occurring within a non-inertial (rotating) frame of reference.

1.4 Co-ordinate systems

In most circumstances fluid motions will be described in Cartesian co-ordinates with x and y locating horizontal co-ordinates and z the vertical. We denote a position in three-dimensional space by $\vec{x} \equiv (x, y, z)$. Velocities are correspondingly denoted by $\vec{u} \equiv (u, v, w)$.

Throughout we use arrows above symbols to denote vectors. The magnitude of the vector is denoted by $|\vec{u}| \equiv (u^2 + v^2 + w^2)^{1/2}$. Vectors with unit length are denoted by a superimposed hat so that $\hat{x} \equiv \vec{x}/|\vec{x}|$, for example, is the unit vector pointing in the x-direction.

To describe the motion of internal gravity waves on the Earth, one could employ spherical co-ordinates, but the motions are confined sufficiently close to the Earth's surface and with a sufficiently small horizontal extent compared to the Earth's radius that a Cartesian co-ordinate system suffices.

In this system, the z-axis is oriented upwards, pointing parallel to the net gravitational force that includes the combined effect of gravitational and centripetal forces on the rotating Earth. This is discussed in more detail in Section 1.10.

It is typical in geophysical fluid dynamics to orient the x-axis in the zonal direction, pointing from west to east, and to orient the y-axis in the meridional direction, pointing from south to north. Likewise, the velocity components u, v and w correspond to the zonal (eastward), meridional (northward) and vertical (upward) velocity, respectively.

In the study of internal gravity waves, the effective change of the Earth's rotation with latitude is ignored and so the distinction between north–south and east–west becomes irrelevant (see Exercises). The only thing that matters when considering the evolution of large-scale, slowly evolving internal gravity waves is the sign and magnitude of the local effects of rotation. This is discussed in more detail in Section 1.10.3. Here it suffices to say that we can arbitrarily orient the x- and y-axes. This is convenient in the consideration of plane waves for which the x-axis can be oriented in the horizontal direction of wave propagation thus rendering their description two- or even one-dimensional in space.

1.5 Lagrangian and Eulerian frames of reference

For conceptual convenience, it is sometimes useful to describe how a fluid evolves by imagining that it is composed of 'fluid parcels', which are infinitesimally small fluid elements. A parcel is not a real object like a molecule. Effectively, it represents the collective properties of fluid of a sufficiently large volume that atomic-level details can be ignored, but not so large that the properties of the parcel vary within its volume. Thus a fluid parcel can be assigned a density, velocity and so on.

The evolution of a fluid parcel can be described mathematically in a frame of reference that moves with the parcel. This is the Lagrangian description of a fluid. We denote the Lagrangian representation of a fluid parcel's properties with an 'L' subscript. For example, $C_L(t;\vec{x}(t))$ is a property (such as the concentration of injected dye, density, horizontal velocity, etc.) of a fluid parcel situated at $\vec{x}(t_0)$ at time t_0 and which has moved to position $\vec{x}(t_1)$ at time t_1, as illustrated in Figure 1.5.

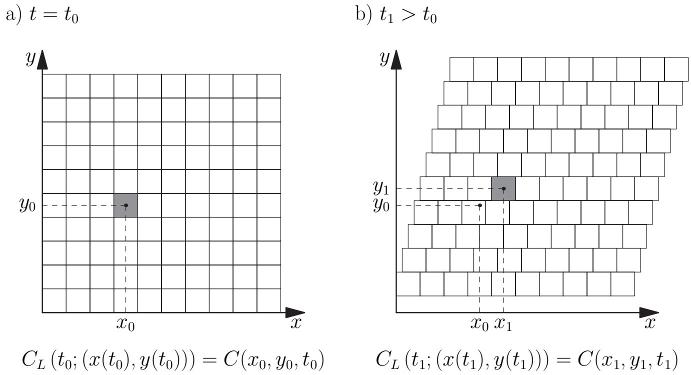

Fig. 1.5. Schematics illustrating the relationship between Eulerian and Lagrangian co-ordinates. A fluid parcel located initially at $(x_0, y_0) \equiv (x(t_0), y(t_0))$, as shown in a), moves to position $(x_1, y_1) \equiv (x(t_1), y(t_1))$ at time t_1, as shown in b). In Lagrangian co-ordinates the properties of this parcel are tracked while it moves; in Eulerian co-ordinates the properties of the fluid passing a fixed position are tracked.

The semicolon in the argument to C_L is meant to indicate that knowing the position of the parcel as time evolves is not necessary in the Lagrangian frame. Only its initial position, $\vec{x}(t_0)$, is necessary for labelling the fluid parcel with which we are dealing at later times. Thus $C_L(t;\vec{x}(t))$ is a function of time alone and so we may denote changes to C_L in the Lagrangian frame by the ordinary time derivative, d/dt.

This is one reason why the Lagrangian description of a fluid is convenient. The statement that a conserved quantity s does not change following the motion of a fluid parcel is given simply by the differential equation

$$\frac{ds_L}{dt} = 0. \tag{1.1}$$

Though appealing for its simple description of conservation laws, the Lagrangian frame is often impractical for analytical and numerical calculations because it does not explicitly specify the differences between a parcel and its surroundings. To work out how density varies spatially at some time, one would have to trace the paths of all surrounding fluid parcels back to time $t = t_0$, when their properties were first prescribed.

The Eulerian frame is often more practical. Rather than focusing upon the evolution of a fluid parcel as it moves with a flow, one examines instead how the properties of the fluid change in time at a fixed position. The changes occur not only because the properties of the parcel vary in time but also because a train of parcels with different properties moves past that point.

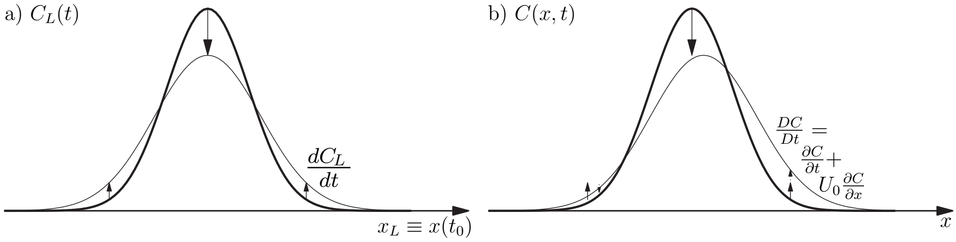

Fig. 1.6. Schematic illustrating the time change of concentration in a) a Lagrangian frame and b) an Eulerian frame in which the diffusing blob advects rightwards at speed U_0. The initial concentration is indicated by the thick solid line and its value a short time later is indicated by the thin line. Solid arrows indicate time variations in concentration due to diffusion. Dotted arrows indicate changes due to advection.

In the Eulerian frame we denote by $C(\vec{x},t)$ the property C of fluid at position $\vec{x}$ and at time t. Here $\vec{x}$ is independent of t and correspondingly we have dropped the L subscript employed in our Lagrangian definition. Translating from the Lagrangian to Eulerian frame is done through straightforward application of the chain rule of (partial) derivatives, which has the effect of switching from a moving to a stationary frame of reference.

To demonstrate this, consider the example in one spatial dimension as illustrated in Figure 1.6. Here $C_L(t)$ represents the concentration, for example of injected dye, associated with fluid parcels located initially at $x(t_0)$. The concentration is assumed to change in time due to diffusion, so that the peak concentration of the central fluid parcel decreases while the concentration of dye associated with parcels at the flanks increases. The rate of change of concentration is given by dC_L/dt. Even if the background flow is moving, in the Lagrangian frame the value of C_L is determined by their initial and not instantaneous position. The value of concentration in both space and time is revealed in the Eulerian frame, shown in Figure 1.6b. In this example, we assume the background flow moves at constant horizontal speed U_0. So, at a fixed position, x, the concentration $C(x,t)$ changes, both because of the instantaneous change in time of concentration $\partial_t C$, and because of the advection of the concentration gradient $U_0 \partial_x C$. Explicitly, after a short time the concentration in the Eulerian frame is related to that in the Lagrangian frame by

$$\frac{d}{dt} C_L(t; x(t_0) + U_0 t) = \frac{\partial C}{\partial t} + U_0 \frac{\partial C}{\partial x}. \qquad (1.2)$$

The concentration changes at a point, both because the concentration of a parcel passing the point changes in time (the first term on the right-hand side of (1.2)), and because the surrounding parcels that move into and away from the point have different concentrations (the second term on the right-hand side of (1.2)).