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## Chaotic Dynamics

An Introduction Based on Classical Mechanics

Since Newton, a basic principle of natural philosophy has been determinism, the possibility of predicting evolution over time into the far future, given the governing equations and starting conditions. Our everyday experience often strongly contradicts this expectation. In the past few decades we have come to understand that even motion in simple systems can have complex and surprising properties.

*Chaotic Dynamics* provides a clear introduction to chaotic phenomena, based on geometrical interpretations and simple arguments, without in-depth scientific and mathematical knowledge. Examples are taken from classical mechanics whose elementary laws are familiar to the reader. In order to emphasise the general features of chaos, the most important relations are also given in simple mathematical forms, independent of any mechanical interpretation. A broad range of potential applications are presented, ranging from everyday phenomena through engineering and environmental problems to astronomical aspects. It is richly illustrated throughout, and includes striking colour plates of the probability distribution of chaotic attractors.

Chaos occurs in a variety of scientific disciplines, and proves to be the rule, not the exception. The book is primarily intended for undergraduate students in science, engineering and mathematics.

TAMÁS TÉL is Professor of Physics at Eötvös University, Budapest, Hungary. He has served on the editorial boards of the journals *Nonlinearity* and *Fractals* and the advisory board of *Chaos*. His main fields of interest are non-linear dynamics, statistical mechanics, fluid dynamics and environmental flows.

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# Chaotic Dynamics

## An Introduction Based on Classical Mechanics

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# Colour plates

**I.** Chaotic attractor of an irregularly oscillating body (a driven non-linear oscillator; Sections 1.2.1 and 5.6.2 and equation (5.85)) on a stroboscopic map (Fig. 1.4), coloured according to the visiting probabilities. The colour change from red to yellow denotes less than 8% of the maximum of the distribution. Between 8 and 30% is depicted by a colour change from yellow to white. Above 30% is represented by pure white. The picture contains 1000 × 1000 points.

**II.** Chaotic attractor of a driven pendulum (Sections 1.2.1 and 5.6.3 and equation (5.89)) on a stroboscopic map (Fig. 1.8), coloured according to the visiting probabilities. Dark green represents up to 4% of the maximum; medium green to yellow represents between 4 and 50%; bright yellow represents 50% and above. The picture contains 1000 × 1000 points.

**III.** Basins of attraction of the three equilibrium states (point attractors marked by white dots) of a magnetic pendulum (Sections 1.2.2 and 6.8.3, Eqs. (6.36) and (6.37)) in the plane of the initial positions parallel to that of the magnets, with zero initial velocities. The friction coefficient is 1.5 times greater than in Fig. 1.10; all the other data are unchanged. The fractal part of the basin boundaries appears to be only slightly extended. The distance between neighbouring initial positions is 1/240 (the resolution is 1280 × 960 points).

**IV.** Basins of attraction of a magnetic pendulum swinging twice as fast as the one in Fig. 1.10 (all the other data are unchanged). The fractal character of the basin boundaries is pronounced.

**V.** Basins of attraction of the magnetic pendulum in Plate IV, but the fixation point of the pendulum is now slightly off the centre of mass of the magnets (it is shifted by 0.2 units diagonally). The strict symmetry of Plate IV has disappeared, but the character has not changed.

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**VII.** Basins of attraction of two stationary periodic motions (limit cycle attractors marked by white dots) of a driven pendulum (Sections 1.2.2 and 5.6.3 and equation (5.89)) on a stroboscopic map (like the one in



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Fig. 1.13). The lighter blue and red hues belong to initial conditions from which a small neighbourhood of the attractors is reached in more than eight periods. The resolution is 1280 × 960 points.

**VIII.** Overview of the frictionless dynamics of a body swinging on a pulley (Sections 1.2.3 and 7.4.3 and equation (7.36)) on a Poincaré map. The mass of the swinging body is smaller than that in Fig. 1.17. The dotted region is a chaotic band, whereas closed curves represent regular, non-chaotic motions. Rings of identical colour become mapped onto each other. Their centres form higher-order cycles (yellow, green and blue correspond, for example, to two-, five- and six-cycles, respectively). The chaotic band pertains to a single, the rings to 24 different, initial condition(s).

**IX.** Mirroring Christmas-tree ornaments (Section 1.2.4). The four spheres touch, and their centres are at the corners of a tetrahedron. The one on the right is red, that on the left is yellow, and the ones behind and on top are silver. The picture shows the reflection pattern of the flash. (Photographed by G. Maros.)

**X.** Christmas-tree ornaments mirrored in each other (Section 1.2.4). All four spheres are now silver, but the surfaces tangent to the spheres are coloured white, yellow and red, and the fourth one (towards the camera) appears to be black. The insets show the reflection images at the centres of the tetrahedron from slightly different views. (Photographed by P. Hámori.)

**XI.** Droplet dynamics in a container with two outlets (Sections 1.2.5 and 9.4.1 and equations (9.24)). The top left picture exhibits the initial configuration of the square-shaped droplet and the state after one time unit (outlets are marked by white dots). The top right, bottom left and bottom right panels display the situation after two, three and four time units, respectively. The part of the droplet that has not yet left the system becomes increasingly ramified, and the colours become well mixed. The picture contains 640 × 480 points.

**XII.** Natural distribution of the roof attractor (the same as that of Fig. 5.44(b) from a different view). The grid used to represent the distribution is of size $\varepsilon = 1/500$. The colour of each column of height up to 2% of the maximum is red, it changes towards yellow up to 50%, and is bright yellow beyond 50%.

**XIII.** Natural distribution on the chaotic attractor of the driven non-linear oscillator presented in Plate I. Blue changing towards red is used up to 10% of the maximum, then red changing towards yellow up to 60%, and bright yellow beyond.

**XIV.** Natural distribution on the chaotic attractor of the driven pendulum presented in Plate II. Dark green is used up to 2% of the maximum,

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then medium green up to 5%, green changhing towards yellow up to 30%, and bright yellow beyond.

**XV.** Natural distribution on the chaotic attractor of an oscillator kicked with an exponential amplitude ($\varepsilon = 1/800$). Same as Fig. P.23(a) from a different view (the distribution is mirroring on the plane of the map).

**XVI.** Natural distribution on the chaotic attractor of an oscillator kicked with a sinusoidal amplitude. Same as Fig. P.23(b) but seen from below ($\varepsilon = 1/200$). Blue changing towards green is used up to 2% of the maximum visible value, then green changing towards yellow up to 20%, and bright yellow beyond.

**XVII.** Natural distribution on the roof saddle (the one presented in Fig. 6.13(c), from a different view). Colouring is the same as in Plate XII.

**XVIII.** The distribution of Plate XVII from the top view and with different colouring.

**XIX.** Natural distribution on the parabola saddle (the one presented in Fig. 6.13(d) but from a different view). Blue is used up to 4% of the maximum visible value, then blue changing to red up to 21%, red changing towards yellow up to 70%, and bright yellow beyond.

**XX.** Natural distribution of a conservative system. The distributions in three chaotic bands (from $10^6$ iterations), coloured in different shades of green, of the kicked rotator (i.e. of the standard map with the parameter of Fig. 7.7(c)) are shown. The distribution is uniform within each band. The numerical convergence towards the smooth distribution is much slower near the edges, which are formed by KAM tori, than in the interior. The yellow barriers represent a few quasi-periodic tori.

**XXI.** Escape regions and boundaries in the three-disc problem; Section 8.2.3. Scattering orbits with the first bounce on the left disc are considered. The initial conditions on the $(\theta_n, \sin\varphi_n)$-plane are coloured according to the deflection angle, $\theta$, of the outgoing orbit: red for $0 < \theta < 2\pi/3$; yellow for $-2\pi/3 < \theta < 0$; and blue for $|\theta| > 2\pi/3$. The escape boundaries contain a fractal component, which is part of the chaotic saddle's stable manifold shown in Fig. 8.12(a). Note that all three colours accumulate along the fractal filaments of the escape boundaries! (A property that also holds for the basin boundaries shown in Plates IV–VI.) The left (right) inset is a magnification of the rectangle marked in the middle (in the left inset), and illustrates self-similarity.

**XXII.** Natural distribution on the attractor of Lorenz's global circulation model (permanent winter, see Section 9.3, Fig. 9.19, from a different view). Colouring is similar to that in Plate XII, but in blue tones.

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**XXIII.** Sea ice advected by the ocean around Kamchatka from the NASA archive (http://eol.jsc.nasa.gov/scripts/sseop/photo.pl?mission=STS045&roll=79&frame=N).

**XXIV.** Plankton distribution around the Shetland Islands, May 12, 2000, from the NASA archive (http://disc.gsfc.nasa.gov/oceancolor/scifocus/oceanColor/vonKarman_vortices.shtml).

**XXV.** Ozone distribution (in parts per million, ppm) above the South Pole region at about 18 km altitude on September 24, 2002. The chemical reactions leading to ozone depletion are simulated over 23 days in the flow given by meteorological wind analyses during a rare event when the so-called 'polar vortex' splits into two parts.[1]

**XXVI.** The same as Plate XXIII but for the distribution of HCl (in parts per billion, ppb). Note that both distributions are filamental but that the concentrations of the two substances are different.[1]

**XXVII (Front cover illustration)** Natural distribution on the chaotic attractor of the non-linear oscillator of Plate XIII with a blueish-white colouring.

**XXVIII (Back cover illustration)** Natural distribution on the chaotic attractor of an oscillator kicked with a parabola amplitude (on the parabola attractor). Colouring is the same as in Plate XIX.

[1] Groop, J.-U., Konopka, P. and Müller, R. 'Ozone chemistry during the 2002 Antarctic vortex split', *J. Atmos. Sci.* **62**, 860 (2005).

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# Preface

> We have just seen that the complexities of things can so easily and dramatically escape the simplicity of the equations which describe them. Unaware of the scope of simple equations, man has often concluded that nothing short of God, not mere equations, is required to explain the complexities of the world.
>
> . . . The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations.
>
> *Richard Feynman in 1963, the year of publication of the Lorenz model*[1]

The world around us is full of phenomena that seem irregular and random in both space and time. Exploring the origin of these phenomena is usually a hopeless task due to the large number of elements involved; therefore one settles for the consideration of the process as noise. A significant scientific discovery made over the past few decades has been that phenomena complicated *in time* can occur in simple systems, and are in fact quite common. In such *chaotic* cases the origin of the random-like behaviour is shown to be the strong and non-linear interaction of the few components. This is particularly surprising since these are systems whose future can be deduced from the knowledge of physical laws and the current state, in principle, with arbitrary accuracy. Our contemplation of nature should be reconsidered in view of the fact that such deterministic systems can exhibit random-like behaviour.

Chaos is the complicated temporal behaviour of simple systems. According to this definition, and contrary to everyday usage, chaos is not spatial and not a static disorder. Chaos is a type of *motion*, or more generally a type of temporal evolution, dynamics. Besides numerous everyday processes (the motion of a pinball or of a snooker ball, the auto-excitation of electric circuits, the mixing of dyes), chaos occurs in technical, chemical and biological phenomena, in the dynamics of illnesses, in

[1] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics, Vol. II.* New York: Addison-Wesley, 1963, Chap. 40, pp. 11, 12.



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elementary economical processes, and on much larger scales, for example in the alternation of the Earth's magnetic axis or in the motion of the components of the Solar System.

There is an active scientific and social interest in this phenomenon and its unusual properties. The motion of chaotic systems is complex but understandable: it provides surprises and presents those who investigate it with the delight of discovery.

Although numerous books are available on this topic, most of them follow an interdisciplinary presentation. The aim of our book is to provide an introduction to the realm of chaos related phenomena within the scope of a single discipline: classical mechanics. This field has been chosen because the inevitable need for a probabilistic view is most surprising within the framework of Newtonian mechanics, whose determinism and basic laws are well known.

The material in the book has been compiled so as to be accessible to readers with only an elementary knowledge of physics and mathematics. It has been our priority to choose the simplest examples within each topic; some could even be presented at secondary school level. These examples clearly show that almost all the mechanical processes treated in basic physics become chaotic when slightly generalised, i.e. when freed of some of the original constraints: chaos is not an *exceptional*, rather it is a *typical* behaviour.

The book is primarily intended for undergraduate students of science, engineering, and computational mathematics, and we hope that it might also contribute to clarifying some misconceptions arising from everyday usage of the term 'chaos'.

The book is based on the material that one of us (T. T.) has been teaching for fifteen years to students of physics and meteorology at Eötvös University, Budapest, and that we have been lecturing together in the last few years.

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# How to read the book

The first part of the book presents the basic phenomena of chaotic dynamics and fractals at an elementary level. Chapter 1 provides, at the same time, a preview of the five main topics to be treated in Part III.

Part II is devoted to the analysis of simple motion. The geometric representation of dynamics in phase space, as well as basic concepts related to instability (hyperbolic points and stable and unstable manifolds), are introduced here. Two-dimensional maps are deduced from the equations of motion for driven systems. Elementary knowledge of ordinary differential equations, of linear algebra, of the Newtonian equation of a single point mass and of related concepts (energy, friction and potential) is assumed.

Part III provides a detailed investigation of chaos. The dynamics occurring on chaotic attractors characteristic of frictional, dissipative systems is presented first (Chapter 5). No preliminary knowledge is required upon accepting that two-dimensional maps can also act as the law of motion. Next, the finite time appearance of chaos, so-called transient chaotic behaviour, is investigated (Chapter 6). Subsequently, chaos in frictionless, conservative systems is considered in Chapter 7, along with its transient variant in the form of chaotic scattering in Chapter 8. Chapter 9 covers different applications of chaos, ranging from engineering to environmental aspects.

Problems constructed from the material of each chapter (many also require computer-based experimentation) motivate the reader to carry out individual work. Some of the solutions are given at the end of the book; the remainder appear (in a password-protected format) on the following website: www.cambridge.org/9780521839129.

Topics only loosely related to the main train of ideas, but of historical or conceptual interest, are presented in Boxes. Some important technical matter (for example numerical algorithms, writing equations in dimensionless forms) are relegated to an Appendix. A bibliography is given at the end of the book, and it is broken down according to topics, chapters and Boxes.

In order to emphasize the general aspects of chaos, the most important relations are also given in a formulation independent of mechanics



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(see Sections 3.5, 4.7, 5.4, 6.3, 7.5 and 8.4). The description of motion occurs primarily in terms of ordinary differential equations, and we concentrate on chaos from such a mathematical background. Irregular dynamics generated by other mathematical structures, which do not represent real phenomena, are thus beyond the scope of the book. The case of one-dimensional maps is mentioned therefore as a special limit only. This approach might provide a useful introduction to chaos for all disciplines whose dynamical phenomena are described by ordinary differential equations.

The book is richly illustrated with computer-generated pictures (24 of which are in colour), not only to provide a better understanding, but also to exemplify the novel and aesthetically appealing world of the geometry of dynamics.

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