Part I
The phenomenon: complex motion, unusual geometry

Chapter 1
Chaotic motion

1.1 What is chaos?

Certain long-lasting, sustained motion repeats itself exactly, periodically. Examples from everyday life are the swinging of a pendulum clock or the Earth orbiting the Sun. According to the view suggested by conventional education, sustained motion is always regular, i.e. periodic (or at most superposition of periodic motion with different periods). Important characteristics of a periodic motion are: (1) it repeats itself; (2) its later state is accurately predictable (this is precisely why a pendulum clock is suitable for measuring time); (3) it always returns to a specific position with exactly the same velocity, i.e. a single point characterises the dynamics when the return velocity is plotted against the position.

Regular motion, however, forms only a *small part* of all possible sustained motion. It has become widely recognised that long-lasting motion, even of simple systems, is often *irregular* and does not repeat itself. The motion of a body fastened to the end of a rubber thread is a good example: for large amplitudes it is much more complex than the simple superposition of swinging and oscillation. No regularity of any sort can be recognised in the dynamics.

The irregular motion of simple systems, i.e. systems containing only a few components, *is called chaotic*. As will be seen later, the existence of such motion is due to the fact that even *simple* equations can have very *complicated* solutions. Contrary to the previously generally accepted view, the simplicity of the equations of motion does not determine whether or not the motion will be regular.

Understanding chaotic motion requires a non-traditional approach and specific tools. Traditional methods are unsuitable for the description

Table 1.1. *Comparison of regular and chaotic motion.*

Regular motion	Chaotic motion
self-repeating	irregular
predictable	unpredictable
of simple geometry	of complicated geometry

of such motion, and the discovery of the ubiquity of chaotic dynamics has become possible through *computer-based experimentation*. Detailed observations have led to the result that chaotic motion is characterised by the *opposite* of the three properties mentioned above: (1) it does not repeat itself, (2) it is unpredictable because of its sensitivity to the initial conditions that are never exactly known, (3) the return rule is complicated: a complex but regular structure appears in the position vs. velocity representation. The differences between the two types of dynamics are summarised in Table 1.1.

The properties of chaotic systems are unusual, either taken individually or together; the most efficient way to understand them is by considering particular cases. In the following, we present the chaotic motion of very simple systems on the basis of numerical simulations, which are unavoidable when studying chaos. It should be emphasised that all of our examples are discussed for a unique set of parameters, and that slightly different choices of the parameters could result in substantially different behaviour. These examples also serve to classify different types of chaos and help in developing the new concepts necessary for a detailed understanding of chaotic dynamics.

1.2 Examples of chaotic motion

1.2.1 Irregular oscillations, driven pendulum – the chaotic attractor

Objects mounted on spring suspensions (for example car wheels and spin-dryers) oscillate. Because of the losses that are always present due to friction or air drag, these oscillations, when left alone, are damped and ultimately vanish. Sustained motion can only develop if energy is supplied from an external source. The supplied energy can be a more or less periodic shaking, i.e. the application of a driving force (caused by interactions with pot-holes in the case of the car wheel and by the uneven distribution of clothes in the spin-dryer), as indicated schematically in Fig. 1.1.

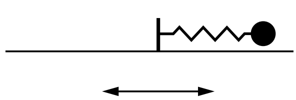

Fig. 1.1. Model of driven oscillations: a body of finite mass is fixed to one end of a weightless spring and the other end of the spring is moved sinusoidally with time.

As long as the displacement is small, the spring obeys a *linear* force law to a good approximation: the magnitude of the restoring force is

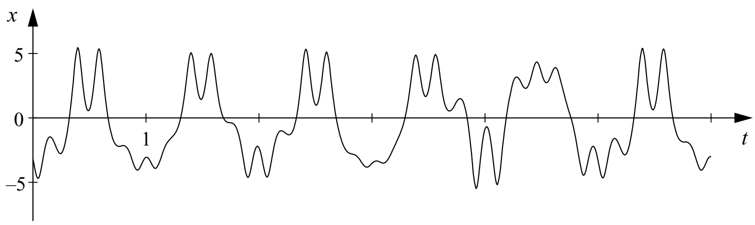

Fig. 1.2. Irregular sustained oscillations of a point mass fixed to the end of a stiffening spring (a driven non-linear oscillator), driven sinusoidally in the presence of friction.

proportional to the elongation. In this case the sustained motion is regular: it adopts the period of the driving force. If the natural period of the spring is close to that of the driving force, then the amplitude may become very large and the well known phenomenon of *resonance* develops. For large amplitudes, however, the force of the spring is usually no longer proportional to the elongation; i.e., the force law is *non-linear*. Resonance is therefore a characteristic example for the appearance of non-linearity.

For non-linear force laws, the restoring force increases more rapidly or more slowly than it would in linear proportion to the elongation: we can speak of stiffening or softening springs, respectively. Whichever type of non-linearity is involved, the sustained state of the driven oscillation may be chaotic. A qualitative explanation is that the spring is not able to adopt exactly the sinusoidal, harmonic motion of the forcing apparatus, since its own periodic behaviour is no longer harmonic. Thus, the sustained dynamics follows the driving force in an averaged sense only, but always differs from it in detail (instead of the uniform hum of the car or the spin-dryer, an irregular sound can be heard in such situations). Neither the amplitude nor the frequency is uniform: the sustained motion does not repeat itself regularly; it is chaotic.

Figure 1.2 shows the motion of a body fixed to the end of a stiffening spring and driven sinusoidally.[1] It can clearly be seen that there is no repetition in the displacement vs. time curve; i.e., the motion is *irregular*.

Slightly different initial conditions result in significant differences in the displacement after only a short time (Fig. 1.3): the dynamics is unpredictable. This figure also shows that the long-term behaviour is of a similar nature in both cases: the two motions are equivalent in a *statistical* sense.

[1] The precise equations of motion of the examples in this section can be found in Sections 5.6.2 and 5.6.3.

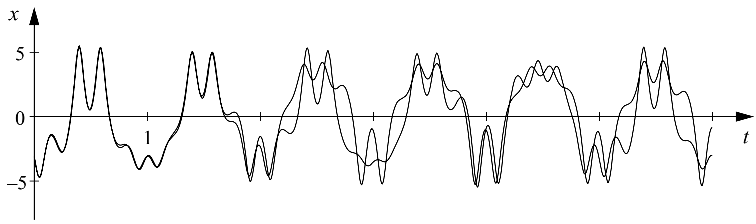

Fig. 1.3. Two sets of motion which started from nearly identical positions. The small initial difference increases rapidly: the motion is sensitive to the initial conditions and therefore it is unpredictable.

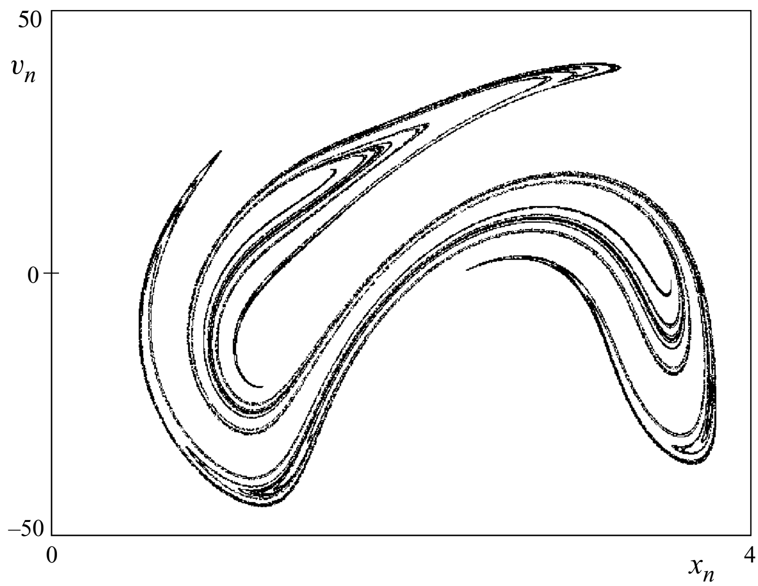

Fig. 1.4. Pattern resulting from a sustained non-linear oscillation in the velocity vs. position representation, using samples taken at time intervals corresponding to the period of the driving force. The position and velocity co-ordinates of the nth sample are x_n and v_n, respectively.

An interesting structure reveals itself when we do not follow the motion continuously, but only 'take samples' of it at equal time intervals. Figure 1.4 and Plate I have been generated by plotting the position and velocity co-ordinates (x_n, v_n) of the sustained motion at integer multiples, n, of the period of the driving force, through several thousands of periods.

It is surprising that there are numerous values of x_n to which many (according to detailed examinations, an *infinite* number of) different velocity values belong. Furthermore, the possible velocity values corresponding to a single position co-ordinate x_n do *not* form a continuous interval anywhere. The whole picture has a thready, filamentary pattern, indicating that chaos is associated with a definite structure. This pattern is much more complicated than those of traditional plane-geometrical objects: it is a structure called a *fractal* (a detailed definition of fractals will be given in Chapter 2). Remember that a *single point* would correspond to a periodic motion in this representation. Chaotic motion is therefore infinitely more complicated than periodic motion.

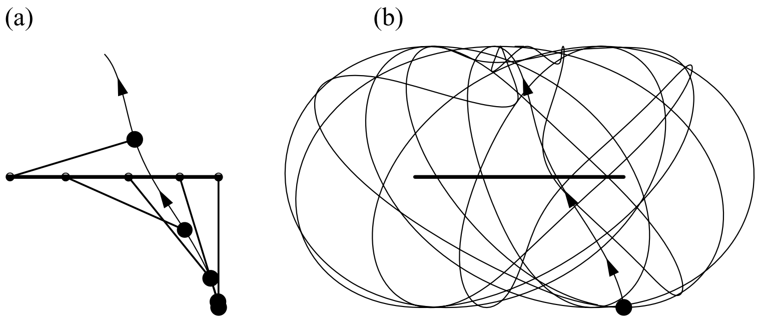

Fig. 1.6. Motion of a driven pendulum. (a) The pendulum a few moments after starting from a hanging state (over the first half period). (b) The path of the end-point of the pendulum for a longer time: the pendulum swings irregularly and often turns over. The horizontal bar indicates the interval over which the suspension point moves.

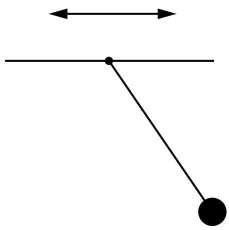

Fig. 1.5. Driven pendulum: the pendulum is driven by the periodic movement of its point of suspension in the horizontal plane.

Another example is the behaviour of a driven pendulum (Fig. 1.5). The large-amplitude swinging of a traditional simple pendulum is non-linear, since the restoring force is not proportional to the deflection angle but to the sine of this angle. Without any driving force, the swinging ceases because of friction or air drag: sustained motion is impossible. The pendulum can be driven in different ways. We examine the case when the point of suspension is moved horizontally, sinusoidally in time. In order to avoid the problem of the folding of the thread, the point mass is considered to be fixed to a very light, thin rod. With a sufficiently strong driving force, the motion may become chaotic. Figure 1.6 shows the path of the pendulum in the vertical plane.

Note that the pendulum turns over several times in the course of its motion. The 'upside down' state is especially unstable, just like that of a pencil standing on its point. Two paths of the pendulum starting from nearby initial positions remain close to each other only until an unstable state, an 'upside down' state, separates them. Then one of them turns over, while the other one falls back to the side it came from (Fig. 1.7). The reason for the unpredictability is that the motion passes through a series of *unstable states*.

The structure underlying the irregular motion can again be demonstrated by following the motion initiated in Fig. 1.6 for a long time and taking samples from it by plotting the position (angular deflection) and velocity (angular velocity) co-ordinates (x_n, v_n) at intervals corresponding to the period of the driving force (Fig. 1.8 and Plate II).

In a frictional (dissipative) system, sustained motion can only develop if some external energy supply (driving) is present. Regardless of the initial state, the dynamics converges to some sustained behaviour that will therefore be called an attracting object, or an *attractor* (for the

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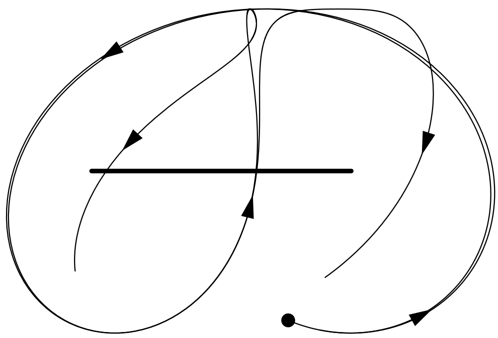


Fig. 1.7. Separation of the paths of two identical driven pendulums starting from nearby points while passing an unstable state. The notation is the same as in Fig. 1.6. The arrows show the direction in which the end-points of the pendulums move.

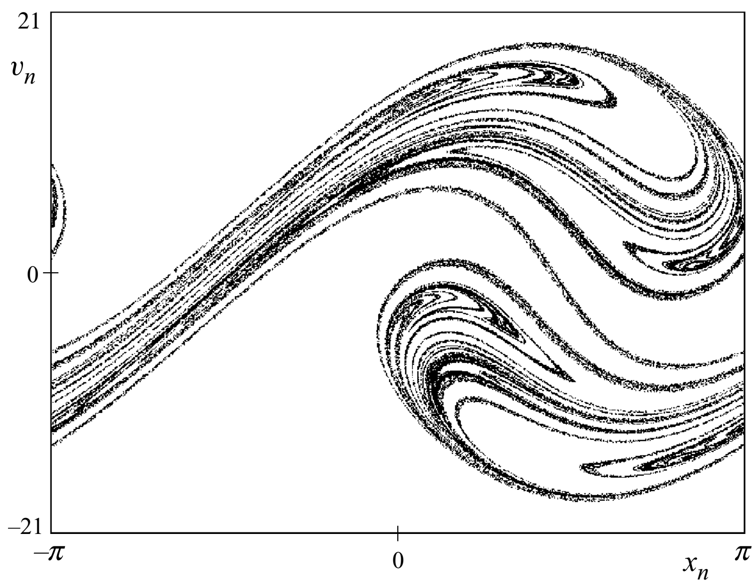


Fig. 1.8. Pattern resulting from a chaotic driven pendulum (chaotic attractor) obtained by plotting the state of the pendulum in the position–velocity co-ordinates at integer multiples of the driving period.

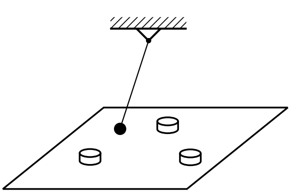


Fig. 1.9. The magnetic pendulum: magnets are fixed to the table and a point mass attracted by the magnets is fixed to the end of the thread. The pendulum ultimately settles in an equilibrium state pointing towards one of the magnets, but only after some irregular, chaotic motion.

exact definition, see Section 3.1.2). *Simple* attractors correspond either to regular or to ceasing motion. A sufficiently large supply of energy inevitably brings about the non-linearity of the system; the sustained dynamics is then usually irregular, i.e. chaotic. This is accompanied by the presence of a *chaotic attractor*, also called a *strange attractor* because of its peculiar structure. Figures 1.4 and 1.8 display examples of chaotic attractors.

1.2.2 Magnetic and driven pendulums, fractal basin boundary – transient chaos

Consider a pendulum, the end-point of which is a small magnetic body, moving above three identical magnets placed at the vertices of a horizontal equilateral triangle (Fig. 1.9). When the force between the

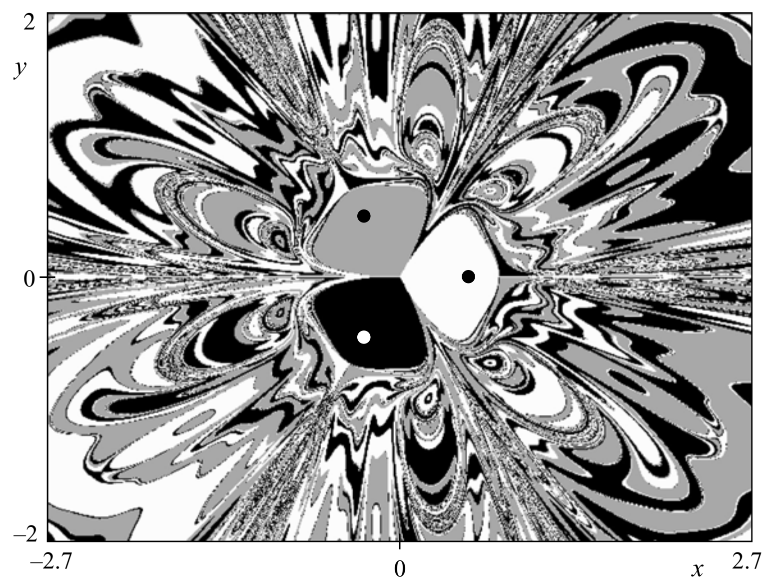


Fig. 1.10. Basin of attraction of the three equilibrium states of the magnetic pendulum (one white and two black dots). Each point on the horizontal plane is shaded according to the magnet in whose neighbourhood the pendulum comes to a rest when starting above that point with zero initial velocity.

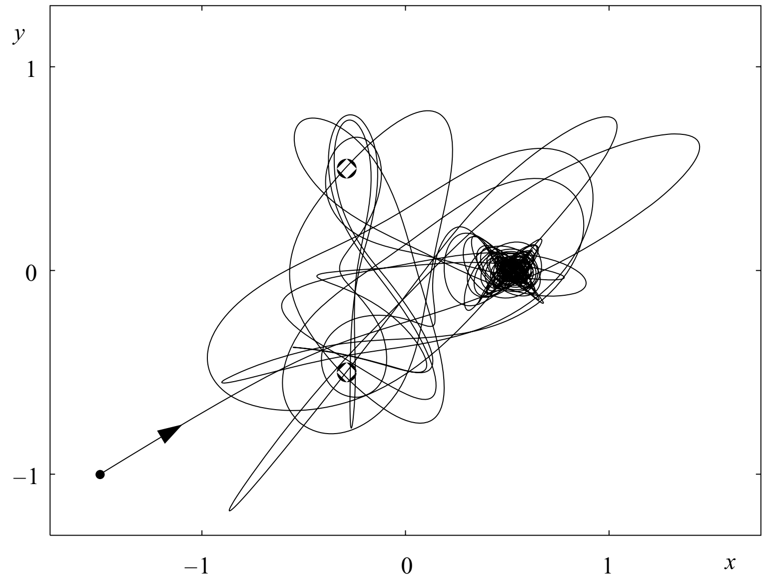
end of the pendulum and the magnets is attracting, the pendulum can come to a halt, pointing towards any of the magnets. Thus there are three simple attractors in the system. Starting above any point of the plane, we can use a computer to calculate which magnet the pendulum will be closest to after coming to rest.[2] By assigning three different colours to the three attractors, and to the corresponding initial positions that converge towards them, the whole plane can be coloured. Each identically coloured area is a *basin of attraction*. Surprisingly, the basin boundaries are interwoven and entangled in a complicated manner (see Fig. 1.10 and Plates III–VI); these simple attractors have *fractal basin boundaries*. (Naturally, the close vicinity of each attractor appears in one colour only: the boundaries do not come close to the attractors.)

Motion starting near the fractal boundary remains irregular for a while, exhibiting *transient chaos*, i.e. chaos lasting for a finite period of time (Fig. 1.11), but ultimately it ends up on one of the attractors.

A driven pendulum (Fig. 1.5) may also exhibit transient chaos. When the friction is sufficiently large, the pendulum can exhibit regular sustained motion only. There are two options for the given parameters (see Fig. 1.12, which depicts the paths corresponding to these two simple attractors in the vertical plane). An overall view of the basins of attraction can again be obtained by representing the starting point in the position

[2] The equations of motion of the magnetic pendulum can be found in Section 6.8.3.

Fig. 1.11. Path of the end-point of the magnetic pendulum viewed from above. The motion is irregular before reaching one of the rest positions: it is transiently chaotic. (The fixed magnets are represented by solid black dots.)



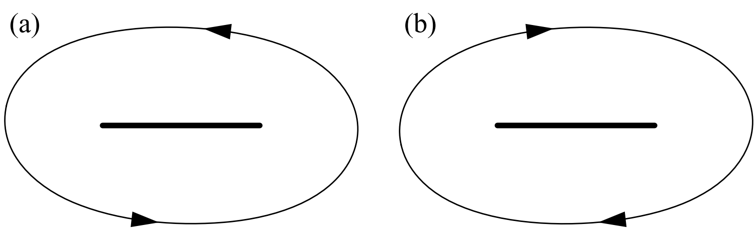


Fig. 1.12. Simple periodic attractors of the driven pendulum: for sufficiently strong friction only these two types of sustained motion exist. All the different initial conditions lead to one of these motions, corresponding to a simple attractor each.

(angular deflection) – velocity (angular velocity) plane in the colour of the attractor which the motion ultimately converges to (Fig. 1.13 and Plate VII).

Motion starting close to the boundary is similar initially to that seen in the case of the chaotic attractor, but it ultimately converges to one of the simple attractors. Irregular dynamics has a finite duration; it is transient. There exist, however, very exceptional initial conditions from which the dynamics never reaches any of the attractors, and is chaotic for any length of time. There exists an infinity of such motion (Fig. 1.14), but the initial conditions that describe these state do not form a compact domain in the plane, but rather a fractal cloud of isolated points called a *chaotic saddle*.

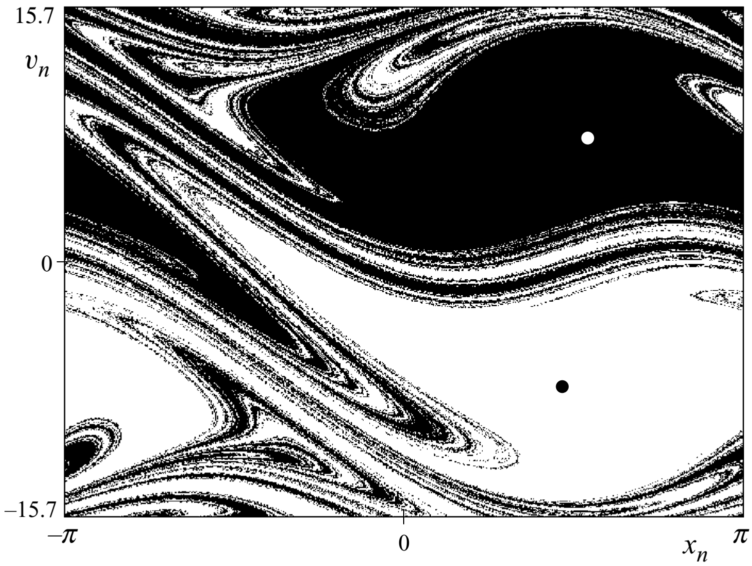


Fig. 1.13. Basins of attraction in the driven pendulum on the plane of initial conditions. The two simple attractors in Fig. 1.12 appear here as points (white and black dots), and the initial states converging towards them are marked in black and white, respectively.

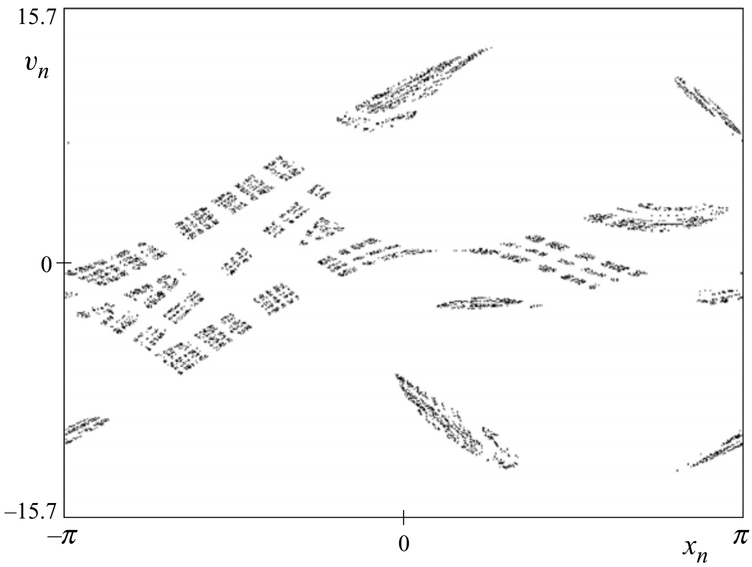


Fig. 1.14. Initial states of the driven pendulum of Fig. 1.13 that never reach either simple attractor: all points shown here are on the basin boundary and, if followed in time, they keep moving between themselves after every period of the driving force. This chaotic saddle is responsible for chaotic dynamics of transient type.

Thus, chaotic dynamics can also occur if the sustained forms of motion are regular, but there are many possible transient routes (chaotic transients) leading to them. In such cases several simple attractors co-exist, each with its own basin of attraction defined by the set of initial conditions which converges to the given attractor. The basins of attraction often penetrate each other, and their boundaries can also be filamentary fractal curves. The motion starting from the vicinity of these fractal basin