EQUIVALENCE AND DUALITY
FOR MODULE CATEGORIES
(with Tilting and Cotilting for Rings)

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Some Module Theoretic Observations

We begin with a chapter consisting of several general facts involving various closure properties of certain categories of modules. These results are part of the background necessary for our future chapters, and we believe that they are of interest in themselves.

Throughout this book $R$ denotes an associative ring with identity $1 \in R$, and $\text{Mod}-R$ and $R\text{-Mod}$ represent the categories of right and left $R$-modules and homomorphisms, while $\text{mod}-R$ and $R\text{-mod}$ denote their subcategories of finitely generated modules.

1.1. The Kernel of $\text{Ext}^1_R(V, \_)$

For any $R$-module $V$ we denote the kernel of $\text{Ext}^1_R(V, \_)$ by $V^\perp$. Closure properties of $V^\perp$ are related to both homological and module-theoretic properties of $V$.

We denote the projective dimension of a module $M$ by $\text{proj} \cdot \dim M$.

**Proposition 1.1.1.** $V^\perp_R$ is closed under factors if and only if $\text{proj} \cdot \dim V_R \leq 1$.

**Proof.** If $V^\perp_R$ is closed under factors, $M \in \text{Mod}-R$, and $E(M)$ is the injective envelope of $M$, then, since $E(M)/M \in V^\perp$, the exactness of the sequence

$$0 = \text{Ext}^1_R(V, E(M)/M) \to \text{Ext}^1_R(V, M) \to \text{Ext}^2_R(V, E(M)) = 0$$

implies $\text{proj} \cdot \dim V_R \leq 1$. Conversely, if $\text{proj} \cdot \dim V_R \leq 1$ and $M \in V^\perp$ with $K$ a submodule of $M$, we obtain $M/K \in V^\perp$ from the exactness of the sequence

$$0 = \text{Ext}^1_R(V, M) \to \text{Ext}^1_R(V, M/K) \to \text{Ext}^2_R(V, K) = 0.$$  

**Proposition 1.1.2.** If $V_R \in \text{Mod}-R$ is finitely presented, then $\text{Ext}^1_R(V, \_)$ commutes with direct sums, so $V^\perp_R$ is closed under direct sums.
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Proof. If \( \{M_\alpha\}_{\alpha \in A} \) is a family of modules in \( \text{Mod-} R \), the natural monomorphism \( \phi_V : \text{Hom}_R(V, M_\alpha) \to \text{Hom}_R(V, \oplus A M_\alpha) \) is an isomorphism whenever \( V \) is finitely generated by [1, Exercise 16.3]. Moreover, \( \phi \) induces natural homomorphisms \( \theta_M : \text{Ext}^1_R(M, M_\alpha) \to \text{Ext}^1_R(M, \oplus A M_\alpha) \). By hypothesis there is an exact sequence \( 0 \to K \to P \to V \to 0 \) with \( P, K \) finitely generated and \( P \) projective. We obtain the commutative diagram with exact rows

\[
\begin{align*}
\oplus A \text{Hom}_R(P, M_\alpha) & \to \oplus A \text{Hom}_R(K, M_\alpha) \to \oplus A \text{Ext}^1_R(V, M_\alpha) \to 0 \\
\downarrow \phi_P & \downarrow \phi_K \downarrow \theta_V \\
\text{Hom}_R(P, \oplus A M_\alpha) & \to \text{Hom}_R(K, \oplus A M_\alpha) \to \text{Ext}^1_R(V, \oplus A M_\alpha) \to 0
\end{align*}
\]

from which the lemma follows. ■

We note that a partial converse of this last result is found in the proof of Lemma 1.2 of [77].

Proposition 1.1.3. If \( V_R \) is finitely generated and \( V_R^\perp \) is closed under factors and direct sums, then \( V_R \) is finitely presented.

Proof. We have \( \text{proj. dim } V_R \leq 1 \) by Proposition 1.1.1; thus, since \( V_R \) is finitely generated, there is an exact sequence \( 0 \to L \to R^n \to V \to 0 \), where \( L \) is projective. Hence, there is a split monomorphism \( j : L \to R^{(X)} \) for some set \( X \). By hypothesis \( E(R^{(X)}) \in V^\perp \), so the composition of \( j \) with the inclusion \( i \) of \( R^{(X)} \) into \( E(R^{(X)}) \) has an extension to an element \( f \in \text{Hom}(R^n, E(R^{(X)})) \). Then \( f(R^n) \subseteq E(R^{(F)}) \subseteq E(R^{(X)}) \) for some finite subset \( F \) of \( X \). It follows that \( j(L) \subseteq R^{(F)} \subseteq R^{(X)} \); therefore, since \( j \) is split monic, \( L \) is finitely generated. ■

1.2. Gen(\( V \)) and Finiteness

We recall (see [1]) that for any collection \( V \) of \( R \)-modules, \( \text{Gen}(V) \) (\( \text{gen}(V) \)) denotes the full category of \( R \)-modules that are epimorphic images of (finite) direct sums of modules isomorphic to those in \( V \), and we let \( \text{Tr}_V(M) \) denote the trace of \( V \) in \( M \), the unique largest submodule of \( M \) that belongs to \( \text{Gen}(V) \). If \( V \) consists of a single module \( V_R(M) \) we simply write \( \text{Gen}(V_R) \), and if \( S = \text{End}(V_R) \), then \( \text{Tr}_V(M) \) is the image of the canonical mapping \( \nu_M : V \otimes S \text{Hom}_R(V, M) \to M \).

In order to characterize when \( \text{Gen}(V_R) \) is closed under direct products, we employ the following notions and lemmas.
1.2. Gen(V) and Finiteness

Given that \{M_\alpha\}_{\alpha \in A} is a family in Mod-\(R\), for each \(N \in R-\text{Mod}\) we let
\[
\eta_{\Pi A M_\alpha, N} : (\Pi A M_\alpha) \otimes_R N \to \Pi A (M_\alpha \otimes_R N)
\]
denote the canonical mapping to obtain a natural transformation
\[
\eta_{\Pi A M_\alpha} : ((\Pi A M_\alpha) \otimes_\_ \_) \to \Pi A (M_\alpha \otimes_\_ \_).
\]

**Lemma 1.2.1.** Suppose that \{M_\alpha\}_{\alpha \in A} is a family in Mod-\(R\). If \(R N\) is finitely generated (finitely presented), then the canonical homomorphism
\[
\eta_{\Pi A M_\alpha, N} : (\Pi A M_\alpha) \otimes_R N \to \Pi A (M_\alpha \otimes_R N)
\]
is an epimorphism (isomorphism).

**Proof.** If \(\{x_1, \ldots, x_n\}\) generate \(R N\), then any element of \(M_\alpha \otimes_R N\) can be written in the form \(\sum m_i x_i\).

Now assume that \(N\) is finitely presented and let \(0 \to K \to P \to N \to 0\) be an exact sequence with \(P\) finitely generated and projective and \(K\) finitely generated. Then we have a commutative diagram
\[
\begin{array}{cccc}
(\Pi A M_\alpha) \otimes K & (\Pi A M_\alpha) \otimes P & (\Pi A M_\alpha) \otimes N & 0 \\
\downarrow \eta_{\Pi A M_\alpha, K} & \downarrow \eta_{\Pi A M_\alpha, P} & \downarrow \eta_{\Pi A M_\alpha, N} & \\
\Pi A (M_\alpha \otimes K) & \Pi A (M_\alpha \otimes P) & \Pi A (M_\alpha \otimes N) & 0
\end{array}
\]
with exact rows, in which \(\eta_{\Pi A M_\alpha, K}\) is epic, and \(\eta_{\Pi A M_\alpha, P}\) is easily seen to be an isomorphism by naturality of \(\eta_{\Pi A M_\alpha}\). Hence, by the Five Lemma, \(\eta_{\Pi A M_\alpha, N}\) is an isomorphism. ■

Identifying \(R \otimes_R N = N\), we have the following result.

**Lemma 1.2.2.** Let \(N \in R-\text{Mod}\). Then the canonical homomorphism
\[
\eta_{R^A, N} : (R^A) \otimes_R N \to N^A
\]
is an epimorphism (isomorphism) for all sets \(A\) if and only if \(N\) is finitely generated (finitely presented).

**Proof.** The condition is sufficient in either case by Lemma 1.2.1. Conversely, letting \(A = N\), if the diagonal element \(n\) is the image of some element \(\sum_{i=1}^m (a_i) x_i\), then, for all \(n \in N, n = \sum_i a_i x_i\). Thus, \(R N\) is finitely generated whenever \(\eta_{R^A, N}\) is epic. Now supposing that \(\eta_{R^A, N}\) is an isomorphism for all sets \(A\), there is an exact sequence \(0 \to K \to P \to N \to 0\)
Some Module Theoretic Observations

with $P$ finitely generated and projective. Then both $\eta_{R^A, P}$ and $\eta_{R^A, N}$ are isomorphisms in the commutative diagram

$$
\begin{array}{cccccc}
R^A \otimes K & \longrightarrow & R^A \otimes P & \longrightarrow & R^A \otimes N & \rightarrow 0 \\
\eta_{R^A, K} & \downarrow & \eta_{R^A, P} & \downarrow & \eta_{R^A, N} & \\
0 & \longrightarrow & K^A & \longrightarrow & P^A & \longrightarrow N^A & \rightarrow 0
\end{array}
$$

with exact rows. Hence, by the Snake Lemma, $\eta_{R^A, K}$ is an epimorphism, and so $K$ is finitely generated. ■

Now we are in position to determine just when $\text{Gen}(V_R)$ is closed under direct products.

**Proposition 1.2.3.** The following statements about a module $V_R$ with $S = \text{End}(V_R)$ are equivalent:

(a) $\text{Gen}(V_R)$ contains $V^A$ for all sets $A$;

(b) $\text{Gen}(V_R)$ is closed under direct products;

(c) $S^V$ is finitely generated.

**Proof.** A module $M_R$ is in $\text{Gen}(V_R)$ if and only if the canonical trace mapping $\nu_M : \text{Hom}_R(V, M) \otimes_S V \rightarrow M$ is epic. For any set $A$ we have the commutative diagram

$$
\begin{array}{cccccc}
\text{Hom}_R(V, V^A) \otimes_S V & \xrightarrow{\cong} & \text{Hom}_R(V, V^A) \otimes_S V \\
\downarrow \nu_{V^A} & & \downarrow = \\
V^A & \xleftarrow{=} & S^A \otimes_S V,
\end{array}
$$

so $(a) \Leftrightarrow (c)$ follows from Lemma 1.2.2.

$(b) \Rightarrow (a)$ is clear. For $(c) \Rightarrow (b)$, assume that $S^V$ is finitely generated and $\{M_a\}_{a \in A}$ belong to $\text{Gen}(V_R)$. Then the composite of the canonical homomorphisms

$$
\text{Hom}_R(V, \Pi_A M_a) \otimes_S V \xrightarrow{\cong} (\Pi_A \text{Hom}_R(V, M_a)) \otimes_S V \\
\xrightarrow{\eta} \Pi_A (\text{Hom}_R(V, M_a) \otimes_S V) \\
\xrightarrow{\Pi_A \nu_{M_a}} \Pi_A M_a
$$

is epic by Lemma 1.2.1 and this composite is $\nu_{\Pi_A M_a}$. ■

Next we obtain a mapping, in addition to the trace map, that determines whether a module belongs to $\text{Gen}(V^A)$. 
1.2. Gen(V) and Finiteness

**Lemma 1.2.4.** Let \( i : R \rightarrow V_R \) be a homomorphism. If
\[
\text{Hom}_R(i, M) : \text{Hom}_R(V, M) \rightarrow \text{Hom}_R(R, M)
\]
is an epimorphism, then \( M \in \text{Gen}(V_R) \).

**Proof.** Let \( \varphi : \text{Hom}_R(R, M) \rightarrow M \) be the canonical isomorphism. Then, by hypothesis, for each \( m \in M \) there is an \( f_m \in \text{Hom}_R(V, M) \) such that \( m = \varphi(\text{Hom}_R(i, M)(f_m)) = \varphi(f_m \circ i) = f_m(i(1)) \), so \( m \in \text{Tr}_V(M) \). \Halmos

Let \( V_R \) be a fixed module with \( S = \text{End}(V_R) \). Let \( \{x_a\}_{a \in A} \) be a generating set for \( S \) \( V \) and define \( i : R \rightarrow V_R^A \) via \( i(r) = (x_a r)_{a \in A} \). Plainly, \( \text{Ker}(i) = \text{Ann}_R(V) \). Thus we have the exact sequence
\[
0 \rightarrow \text{Ann}_R(V) \rightarrow R \rightarrow i \rightarrow V_R^A \rightarrow (V^A / i(R))_R \rightarrow 0.
\]
For any \( M \in \text{Mod-R} \), denote by \( i_M^* \) the composite of the homomorphisms
\[
\text{Hom}_R(V^A, M) \rightarrow \text{Hom}_R(R, M) \rightarrow M.
\]

**Proposition 1.2.5.** \( M \in \text{Gen}(V_R^A) \) if and only if \( i_M^* \) is epic. In particular, if \( V_R \) is finitely generated over its endomorphism ring (and \( A \) is taken to be finite), then \( i_M^* \) is epic if and only if \( M \in \text{Gen}(V_R) \).

**Proof.** Denote the class of \( M \in \text{Mod-R} \) for which \( i_M^* \) is epic by \( \mathcal{E} \). We first claim that \( V_R \in \mathcal{E} \) and \( \mathcal{E} \) is closed under epimorphic images. If \( v \in V \), let \( v = \sum_{j=1}^k x_{a_j} v_{a_j} \). Define \( f \in \text{Hom}_R(V^A, V) \) via \( f((v_{a_j})) = \sum_{j=1}^k x_{a_j} v_{a_j} \). Then \( i_M^*(f) = (f \circ i)(1) = v \). For the second assertion suppose that \( M \rightarrow L \) is epic in \( R \)-Mod where \( i_M^* \) is also epic. Then, since \( \eta \circ i_M^* = i_L^* \circ \text{Hom}(V^A, \eta) \), \( i_L^* \) is also epic and our claim is proved.

Next we note that \( \mathcal{E} \) is closed under arbitrary direct sums and products. Plainly, \( \mathcal{E} \) is closed under arbitrary direct products and hence under finite direct sums. Let \( M = \bigoplus_{\beta \in B} M_\beta \), where each \( M_\beta \in \mathcal{E} \), and let \( m \in M \). There is a finite subset \( B_0 \) of \( B \) such that if \( \kappa \) is the canonical inclusion \( M_0 = \bigoplus_{\beta \in B_0} M_\beta \hookrightarrow M \), \( m = \kappa(m') \), with \( m' \in M_0 \). But then there exists \( f \in \text{Hom}_R(V^A, M_0) \) such that \( m = \kappa(m') = \kappa(i_M^*(f)) = i_M^*(\text{Hom}(V^A, \kappa)(f)) \).

Now, \( \text{Gen}(V^A) \subseteq \mathcal{E} \) follows from what we have proved thus far. The reverse inclusion follows immediately from Lemma 1.2.4. \Halmos

Since the class of modules \( M \) for which \( i_M^* \) is epic is clearly closed under direct products, Proposition 1.2.5 implies that \( \text{Gen}(V^A) \) is closed under direct products; hence, we have the following corollary by Proposition 1.2.3.
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Corollary 1.2.6. For any \( V_R \), if \( V \) is generated over its endomorphism ring by \(|A|\) elements, then \( V^A \) is finitely generated over its endomorphism ring.

A module \( V_R \) is small if, as is the case for a finitely generated module, \( \text{Hom}_R(V, \bigoplus_A M_\alpha) \cong \bigoplus_A \text{Hom}_R(V, M_\alpha) \), canonically, for all \( \{M_\alpha\}_A \) in Mod-\( R \). A module \( V_R \) is self-small if \( \text{Hom}_R(V, V(A)) \cong \text{Hom}_R(V, V(A)) \), canonically, for all sets \( A \). This notion is a key element in the proof of the following proposition due to J. Trlifaj [78].

Proposition 1.2.7. If \( \text{Hom}_R(V, \_\ ) \) commutes with direct limits (with directed index sets) of modules in Gen(\( V_R \)), then \( V_R \) is finitely generated.

Proof. First we note that, since \( V(A) = \varinjlim V^{(F)} \) such that \( F \) is a finite subset of \( A \) [69, pp. 44–45], we have, by hypothesis, \( \text{Hom}_R(V, V^{(A)}) = \text{Hom}_R(V, \varinjlim V^{(F)}) \cong \varinjlim \text{Hom}_R(V, V^{(F)}) \cong \text{Hom}_R(V, V^{(A)}) \); thus, \( V \) is self-small. Let \( V = \sum_A x_\alpha R \) and let \( t_\alpha : V \to V^{(A)} (\alpha \in A) \) be the canonical injections. Then, identifying \( x_\alpha = t_\alpha x_\alpha \), since \( V \cong (\bigoplus_A t_\alpha x_\alpha R)/K \), we have a monomorphism

\[
\varphi : V \to V^{(A)}/K
\]

with \( K \leq \bigoplus_A x_\alpha R \) and

\[
\varphi : x_\alpha \mapsto x_\alpha + K.
\]

Then, letting \( \{K_i\}_{i \in I} \) denotes the finitely generated submodules of \( K \), with canonical epimorphisms \( \gamma_i : V^{(A)}/K_i \to V^{(A)}/K (i \in I) \),

\[
(V^{(A)}/K, \{\gamma_i\}) = \varinjlim V^{(A)}/K_i.
\]

Now, by hypothesis,

\[
(\text{Hom}_R(V, V^{(A)}/K), \{\text{Hom}_R(V, \gamma_i)\}_{i \in I}) = \varinjlim \text{Hom}_R(V, V^{(A)}/K_i)
\]

so that (see [69, Theorem 2.17])

\[
\text{Hom}_R(V, V^{(A)}/K) = \bigcup_I \text{Im} \text{Hom}_R(V, \gamma_i).
\]

Thus there is an \( i \in I \) and a \( \varphi_i \in \text{Hom}_R(V, V^{(A)}/K_i) \) with

\[
\varphi = \gamma_i \varphi_i.
\]

There is a finite set \( F \subseteq A \) such that \( K_i \subseteq V^{(F)} \), and hence

\[
V^{(A)}/K_i = V^{(F)}/K_i \oplus V^{(A\setminus F)}.
\]
1.2. Gen($V$) and Finiteness

So since $V$ is self-small

$$\text{Im } \varphi_i \subseteq V^{(H)}/K_i$$

for some finite set $H$ with $F \subseteq H \subseteq A$. Now we have

$$\varphi(V) \subseteq \gamma_i(V^{(H)}/K_i) = (V^{(H)} + K)/K,$$

so, for each $\alpha \in A$, there is a $v_\alpha \in V^{(H)}$ such that

$$v_\alpha + K = \varphi(x_\alpha) = x_\alpha + K.$$

But then

$$v_\alpha \in V^{(H)} \cap (\oplus_A x_\alpha R) = \oplus_H x_\alpha R,$$

and we have

$$\text{Im } \varphi \subseteq (\oplus_H x_\alpha R + K)/K \subseteq \text{Im } \varphi;$$

thus, $V \cong \text{Im } \varphi$ is finitely generated. ■

Another closure property of Gen($V_R$) forces $V$ to be flat over its endomorphism ring.

**Proposition 1.2.8.** Suppose $V \in \text{Mod-}R$ and $S = \text{End}(V_R)$. If Gen($V_R$) is closed under submodules, then $S V$ is flat.

**Proof.** Recall [1, Lemma 19.19] that $S V$ is flat if and only if for every relation

$$\sum_{i=1}^{m} s_i x_i = 0 \quad (s_i \in S, \ x_i \in V)$$

there exist $y_j \in V, \sigma_{ij} \in S, \ 1 \leq i \leq m, \ 1 \leq j \leq n$, such that for all $1 \leq i \leq m$ and $1 \leq j \leq n$

$$\sum_{j=1}^{n} \sigma_{ij} y_j = x_i \text{ and } \sum_{i=1}^{m} s_i \sigma_{ij} = 0.$$ 

So suppose we do have

$$\sum_{i=1}^{m} s_i x_i = 0 \quad (s_i \in S, \ x_i \in V);$$

let $\pi_j : V^m \to V, \ 1 \leq j \leq m$, be the canonical projections and let

$$K = \text{Ker } d$$
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where $d$ is the homomorphism $d : V^{(m)} \rightarrow V$ defined by

$$d : z \mapsto \sum_{i=1}^{m} s_i \pi_i(z), \quad z \in V^{(m)}.$$  

Then $x = (x_1, \ldots, x_m) \in K$ and so, since $V$ generates $K$, there exist

$$f_j : V \rightarrow K, \quad \text{and} \quad y_j \in V, \quad 1 \leq j \leq n$$

such that

$$x = \sum_{j=1}^{n} f_j y_j.$$  

Now let

$$\sigma_{ij} = \pi_i f_j \in S, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

to obtain

$$\sum_{j=1}^{n} \sigma_{ij} y_j = \pi_j(x) = x_i, \quad 1 \leq i \leq m,$$

and for each $u \in V$

$$\sum_{i=1}^{m} s_i \sigma_{ij} u = \sum_{i=1}^{m} s_i \pi_i f_j u = d(f_j u) = 0, \quad 1 \leq j \leq n. \quad \blacksquare$$

### 1.3. Add($V_R$) and Prod($V_R$)

We denote the subcategories of Mod-$R$ consisting of all direct summands of a direct sum, respectively, a direct product, of copies of a module $V_R$ by $Add(V_R)$, respectively, by $Prod(V_R)$.

According to [1, Theorems 19.20 and 28.4], if $S$ is a left coherent right perfect ring, then every direct product of projective right $S$-modules is projective, that is, belongs to $Add(S_S)$. (This result and its converse are due to S. Chase [12], who also proved that if every direct product of copies of $S_S$ is projective, then $S$ is a left coherent right perfect ring.) On the other hand we have

**Lemma 1.3.1.** If $S$ is a left coherent right perfect ring, then every projective right $S$-module belongs to $Prod(S_S)$.

**Proof.** Letting $J = J(S)$, suppose that $P_S$ is projective and $P/PJ = \oplus_{a \in A} T_a$ with each $T_a$ simple. Let $Q = S_S^A$. Then $QJ \leq J^A$ and $\oplus_{a \in A} T_a$ is isomorphic.
1.4. Torsion Theory

Let $V_R$ be a self-small module with $\text{End}(V_R) = S$. If $S$ is left coherent and right perfect, and $S \cdot V$ is finitely presented, then $\text{Prod}(V_R) = \text{Add}(V_R)$.

Proof. Since $V$ is self-small

$$\text{Hom}_R(V, -) : \text{Add}(V_R) \rightleftharpoons \text{Add}(S) : (\cdot \otimes_S V)$$

is an equivalence of categories. But $\text{Hom}_R(V, -)$ commutes with direct products and, by Lemma 1.2.1, so does $(\cdot \otimes_S V)$. Thus the proposition follows from the fact that $\text{Add}(S) = \text{Prod}(S)$. ■

A ring $R$ is an artin algebra if its center $K$ is an artinian ring and $R$ is finitely generated as a $K$-module. Any finitely generated module over an artin algebra is finitely generated over its endomorphism ring, which is also an artin algebra. Thus we have

Corollary 1.3.3. If $V_R$ is a finitely generated module over an artin algebra $R$, then $\text{Prod}(V_R) = \text{Add}(V_R)$.

Note that we have only used one implication of Chase’s theorem. Using his full theorem, H. Krause and M. Saorín showed in [53] that a self-small module $V_R$ with $S = \text{End}(V_R)$ has $\text{Add}(V_R)$ closed under direct products if and only if $S$ is a left coherent right perfect ring and $S \cdot V$ is finitely presented. In view of Proposition 1.3.2 this is equivalent to $\text{Prod}(V_R) = \text{Add}(V_R)$.

1.4. Torsion Theory

Definition 1.4.1. If $\mathcal{C}$ is an abelian category, a torsion theory in $\mathcal{C}$ is a pair of classes of objects $(T, \mathcal{F})$ of $\mathcal{C}$ such that

1. $T = \{ T \in \mathcal{C} \mid \text{Hom}_{\mathcal{C}}(T, F) = 0 \text{ for all } F \in \mathcal{F} \}$,
2. $\mathcal{F} = \{ F \in \mathcal{C} \mid \text{Hom}_{\mathcal{C}}(T, F) = 0 \text{ for all } T \in T \}$,
10 Some Module Theoretic Observations

(3) for each $X \in \mathcal{C}$ there is a subobject $T$ of $X$ such that

$$T \in \mathcal{T} \text{ and } X/T \in \mathcal{F}. $$

When this is the case, the objects in $\mathcal{T}$ are called torsion objects, the elements of $\mathcal{F}$ are called torsion-free objects, and if the object $T$ of (3) is unique, we denote it by $\tau(X)$ and call it the torsion subobject of $X$.

Suppose that $\mathcal{C}$ is a full subcategory of $\text{Mod}-R$ that is closed under submodules, epimorphic images, extensions, direct sums, and direct products. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, then it follows that $\mathcal{T}$ is closed under epimorphic images and direct sums, $\mathcal{F}$ is closed under submodules and direct products, and both are closed under extensions. A class $\mathcal{T}$ ($\mathcal{F}$) of modules in $\mathcal{C}$ with these closure properties is called a torsion (torsion-free) class in $\mathcal{C}$. Then one easily verifies

**Proposition 1.4.2.** Let $\mathcal{C}$ be a full subcategory of $\text{Mod}-R$ that is closed under submodules, epimorphic images, extensions, direct sums, and direct products.

1. If $\mathcal{T}$ is a torsion class in $\mathcal{C}$, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, where $\mathcal{F} = \{ F \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(T, F) = 0 \text{ for all } T \in \mathcal{T} \}$.

2. If $\mathcal{F}$ is a torsion-free class in $\mathcal{C}$, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$, where $\mathcal{T} = \{ T \in \mathcal{C} \mid \text{Hom}_\mathcal{C}(T, F) = 0 \text{ for all } F \in \mathcal{F} \}$.

Dual to $\text{Gen}(\mathcal{V})$, if $\mathcal{V}$ is a class of $R$-modules, $\text{Cogen}(\mathcal{V})(\text{cogen}(\mathcal{V}))$ consists of the $R$-modules that embed in (finite) direct products of modules isomorphic to members of $\mathcal{V}$, and the reject of $\mathcal{V}$ in $M$ is $\text{Rej}_\mathcal{V}(M)$, the intersection of the kernels of all maps from $M$ into members of $\mathcal{V}$.

**Proposition 1.4.3.** Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in $\text{Mod}-R$ and $M$ a module in $\text{Mod}-R$. Then

$$\text{Tr}_\mathcal{T}(M) = \text{Rej}_\mathcal{F}(M).$$

**Proof.** That $\text{Tr}_\mathcal{T}(M) \subseteq \text{Rej}_\mathcal{F}(M)$ follows from $\text{Hom}_R(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$. But since $\text{Tr}_\mathcal{T}(M) \in \mathcal{T}$ and $\mathcal{T}$ is closed under extensions, $M/\text{Tr}_\mathcal{T}(M) \in \mathcal{F}$ and hence $\text{Rej}_\mathcal{F}(M) \subseteq \text{Tr}_\mathcal{T}(M)$. ■

If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\text{Mod}-R$, we let

$$\tau_\mathcal{T}(M) = \text{Tr}_\mathcal{T}(M) = \text{Rej}_\mathcal{F}(M).$$
and call it the torsion submodule of $M$. Then every module in $\text{Mod-R}$ admits an exact sequence

$$0 \rightarrow \tau_T(M) \rightarrow M \rightarrow M/\tau_T(M) \rightarrow 0$$

with $\tau_T(M)$ simultaneously the largest submodule of $M$ belonging to $T$ and the smallest submodule of $M$ such that $M/\tau_T(M)$ belongs to $\mathcal{F}$.

We shall meet torsion theories like those in the following proposition in later sections.

**Proposition 1.4.4.** If $\text{Gen}(V_R) \subseteq V^\perp$, then $(\text{Gen}(V_R), \text{Ker} \text{Hom}_R(V, \_))$ is a torsion theory in $\text{Mod-R}$.

**Proof.** If $0 \rightarrow M_1 \rightarrow X \rightarrow M_2 \rightarrow 0$ is exact with $M_1, M_2 \in \text{Gen}(V_R) \subseteq V^\perp$, letting $S = \text{End}(V_R)$, we obtain a commutative diagram with exact rows

$$\begin{array}{c}
\text{Hom}_R(V, M_1) \otimes_S V \rightarrow \text{Hom}_R(V, X) \otimes_S V \rightarrow \text{Hom}_R(V, M_2) \otimes_S V \rightarrow 0 \\
\nu_{M_1} \downarrow \quad \nu_X \downarrow \quad \nu_{M_2} \downarrow \\
0 \rightarrow M_1 \rightarrow X \rightarrow M_2
\end{array}$$

in which the trace maps $\nu_{M_1}$ and $\nu_{M_2}$ are epimorphisms. But then, by the Snake Lemma, so is $\nu_X$, and Proposition 1.4.2 applies. $\blacksquare$