1

Biography

The known details of the personal side of Frege's life are few.[1] Friedrich Ludwig Gottlob Frege was born November 8, 1848, in Wismar, a town in Pomerania. His father, Karl Alexander (1809–1866), a theologian of some repute, together with his mother, Auguste (d. 1878), ran a school for girls there. Our knowledge of the remainder of Frege's personal life is similarly impoverished. He married Margarete Lieseberg (1856–1904) in 1887. They had several children together, all of whom died at very early ages. Frege adopted a child, Alfred, and raised him on his own.[2] Alfred, who became an engineer, died in 1945 in action during the Second World War.[3] Frege himself died July 26, 1925, at age seventy-seven.

We can say somewhat more about his intellectual life. Frege left home at age twenty-one to enter the University at Jena. He studied mathematics for two years at Jena, and then for two more at Göttingen, where he earned his doctorate in mathematics in December 1873 with a dissertation, supervised by Ernst Schering, in geometry. Although mathematics was clearly his primary study, Frege took a number of courses in physics and chemistry, and, most interestingly for us, philosophy. At Jena, he attended Kuno Fischer's course on Kant's Critical Philosophy, and in his first semester at Göttingen, he attended Hermann Lotze's course on the Philosophy of Religion. The influence and importance of Kant is evident throughout Frege's work, that of Lotze's work on logic is tangible but largely circumstantial.[4]

After completing his *Habilitationsschrift* on the theory of complex numbers, Frege returned to Jena in May of 1874 in the unsalaried position of lecturer [*Privatdozent*]. The position was secured for him by the mathematician Ernst Abbé, his guardian angel at Jena from the time he arrived as a student to his ultimate honorary professorship.[5] Abbé controlled the Carl Zeiss foundation, which received almost half of all the profits from the Zeiss lens and camera factory (which Abbé had helped the Zeiss family establish). Frege's unsalaried honorary professorship at Jena was made possible because he received a stipend from the Zeiss foundation.

Frege taught mathematics at Jena and his first published writings were mainly reviews of books on the foundations of mathematics. In 1879, five years after returning to Jena, he published his *Begriffsschrift*. It was not well received. For one thing, the notation was extraordinarily cumbersome and difficult to penetrate. Also Frege failed to mention, and contrast with his own system, the celebrated advances in logic by Boole and Schröder, in which both classical truth-functional logic and the logic of categorical statements were incorporated into a single mathematical system. In his review of *Begriffsschrift*, Schröder ridiculed the idiosyncratic symbolism as incorporating ideas from Japanese, and as doing nothing better than Boole and many things worse. Schröder had not realized how far Frege had penetrated, and neither did many of his contemporaries.[6]

For three years, Frege worked hard to explain and defend his *Begriffsschrift*, though not with much success.[7] The fault lies in no small measure with Frege himself, for he failed to distinguish in importance the specifics of his notation (which has, thankfully, been totally abandoned) from the logical syntax and semantics it instantiated. What Frege had created, of course, was a formal language in which he axiomatized higher-order quantificational logic; derived many theorems of propositional logic, first-order logic, and second-order logic; and defined the ancestral relation. *Begriffsschrift* represents a milestone, not only in the history of logic and, thereby, in the history of philosophy, but also in the history of modern thought, for it was one of the first sparks in a hundred-year explosion of research into the foundations of mathematics, and into the application of mathematical representation to structures other than numbers and shapes.

Frege soon broke away from this engagement and returned to his creative project announced in *Begriffsschrift*:

> [We] divide all truths that require justification into two kinds, those whose proof can be given purely logically and those whose proof must be grounded on empirical facts. . . . Now, in considering the question of to which of these two kinds arithmetical judgments belong, I first had to see how far one could get in arithmetic by inferences alone, supported only by the laws of thought that transcend all particulars. The course I took was first to seek to reduce the concept of ordering in a series to that of *logical* consequence, in order then to progress to the concept of number. . . . (Frege 1879: 48)

Having codified the notion of proof, of logical consequence, and of ordering in a sequence in *Begriffsschrift*, Frege pursued his investigation into the notion of *cardinal number*, publishing his philosophical strategy in 1884 in *Grundlagen*. Unlike his *Begriffsschrift*, *Grundlagen* is almost devoid of formal symbolism and is otherwise directly engaged with the main views current about arithmetic. His polemic against contemporary empiricist and naturalist views of the concept of number is devastating. It is not only the specifics of these views that Frege believes to be wrong, but also the methodology of seeking a foundation for mathematics by identifying referents for the number words, whether they be material objects, psychological ideas, or Kantian intuitions. This is the cash value of his injunction against looking for the meaning of number words *in isolation*. The numbers, along with sets and the truth values, are *logical objects*: their meaning is intimately bound up with our conceptualization of things. He codified this attitude in his famous *Context Principle* – never to look to the meaning of a word in isolation, but only in the context of a proposition. For Frege, the foundations of mathematics were to be found in the new logic he had created, the language of which was adequate to express all elementary arithmetic statements, so that the truths of logic could be seen to be, when spelled out, truths of logic. *Grundlagen* is widely regarded as a masterpiece written by a philosopher at the height of his powers: in the years from 1884 through the publication of *Grundgesetze*, in 1893, we see Frege at his creative height.

Frege's *Grundlagen*, although free from the symbolism of his more technical works, did not receive much notice, and the little it did receive was, as usual, full of misconceptions. It is not entirely clear why this is so. Perhaps Frege appeared too philosophical for the mathematicians who were working in related areas – he was ignored by Dedekind, roundly criticized by Cantor, and dismissed by Hilbert – and too technical for the philosophers. Only the direct interaction with Husserl – Frege (1894) demolished Husserl's early psychologism in a review – had a clear and immediate impact on active philosophers of his day. Husserl abandoned his psychologism shortly thereafter, but he was none too generous in later life when he recalled Frege to be a man of little note who never amounted to much.

Frege's own philosophical education and his knowledge of historical and contemporary philosophers is extremely problematic. When he quotes from some of the classical philosophers like Descartes, Hobbes, and Leibniz, it is frequently from a popular anthology put together by Baumann (1868) of writings on the philosophy of space and time. Kant gets a great many footnotes, though largely for his work on arithmetic and geometry. It is never clear how much of a philosopher's work Frege was familiar with because he picked and chose discussions that were directly related to the problems he was working on. As with an autodidact, there appear to be immense holes in Frege's knowledge of the history of philosophy; this, plus the single-mindedness with which he approached issues, as if with blinders to what was irrelevant, just underscored his intellectual isolation.

Grundlagen could not, of course, represent the end of his project. Frege would never be satisfied until he demonstrated his position formally. And it was the effort to formalize his view that forced significant changes in the *Grundlagen* story. Frege had tried to make do earlier in *Begriffsschrift* without the notion of set; he had yet to convince himself that the notion was legitimate and that it belonged in logic. At any rate, with the publication of *Grundlagen*, Frege's course was clear: to fill in the logical details of the definition of number he there presented in the manner of his *Begriffsschrift*. What had been missing was a conception of a set; this Frege won through to. Along the way, a sharpening of his philosophical semantics led to the mature views in philosophy of language for which he has been justly celebrated. "Über Sinn und Bedeutung" was published in 1892, and its companion essays appeared in print about that same time.

Grundgesetze was published in 1893 by Hermann Pohle, in Jena. Frege had had difficulty finding a publisher for the book, after the poor reception given to his other works. Pohle agreed to publish the work in two parts: if the first volume was received well, he would publish the second one. Unfortunately it was not received well, to the extent that it was acknowledged by anyone at all. Pohle refused to publish the second volume, and Frege paid for its publication out of his own pocket some ten years later.

Just as Volume 2 of *Grundgesetze* was going to press in 1902, Russell communicated to Frege the famous contradiction he had discovered. Here is the beginning of the first letter to Frege, dated June 16, 1902:

Dear Colleague,
I have known your *Basic Laws of Arithmetic* for a year and a half, but only now have I been able to find the time for the thorough study I intend to devote to your writings. I find myself in full accord with you on all main points, especially in your rejection of any psychological element in logic and in the value you attach to a conceptual notation for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. On many questions of detail, I find discussions, distinctions and definitions in your writings for which one looks in vain in other logicians. On functions in particular (sect. 9 of your *Conceptual*

Notation) I have been led independently to the same views even in detail. I have encountered a difficulty only on one point. You assert (p. 17) that a function could also constitute the indefinite element. This is what I used to believe, but this view now seems to me dubious because of the following contradiction: Let w be the predicate of being a predicate which cannot be predicated of itself. Can w be predicated of itself? From either answer follows its contradictory. We must therefore conclude that w is not a predicate. Likewise, there is no class (as a whole) of those classes which, as wholes, are not members of themselves. From this I conclude that under certain circumstances a definable set does not form a whole. (Frege 1980: 130–1)

From his Axiom 5,

$$\{x|Fx\} = \{x|Gx\} \equiv (\forall x)(Fx \equiv Gx),$$

which lays out the identity conditions for sets, Frege (1893) derives Proposition 91:

$$Fy \equiv y \epsilon \{x|Fx\}.$$

Russell's contradiction is immediate when, in this proposition, the property F is taken to be *is not an element of itself* and the object y is taken to be *the set of all sets that are not elements of themselves*:[8]

$$\neg\{x|\neg x \epsilon x\} \epsilon \{x|\neg x \epsilon x\} \equiv \{x|\neg x \epsilon x\} \epsilon \{x|\neg x \epsilon x\}.$$

Unlike Peano, to whom Russell had also communicated the paradox, Frege acknowledged it with his deep intellectual integrity and attempted to deal with it in an appendix – but to no avail, as he himself acknowledged. He was deeply shaken by this contradiction, which emerged from an axiom about which he had, as he said, always been somewhat doubtful. His life's work in a shambles, Frege's creative energies withered. The foundational paradoxes became a source of immense intellectual stimulation (as Frege himself had surmised in a letter to Russell) and his achievements were soon surpassed by the work of Ernst Zermelo and others. By the time the young Ludwig Wittgenstein came to see him in 1911 to study foundations of mathematics, Frege referred him to Russell. There was a brief flurry of activity in 1918–19 when Frege published some work in philosophy of logic in an Idealist journal. They appear to represent the first chapters of a planned book on logic. These essays remain among the most influential writings of the twentieth century. But the foundations of arithmetic are a different story. We find him saying, in the early 1920s, that he doubts whether sets exist at all. And he is trying to see if the roots of arithmetic are to be found in geometry, a complete turnaround from his earlier views.

That we know of Frege today is largely through his influence on the giants of modern analytic philosophy. Russell was the first to become aware of his work in the philosophy of language and logic. He included an appendix describing Frege's views in his *Philosophy of Mathematics* of 1903. Indeed, immediately afterward, Russell appears to have been most deeply preoccupied with working out Frege's sense/reference theory, an enterprise he abandoned because he thought there were insuperable difficulties with the view and also because he had an alternative in his theory of descriptions. Wittgenstein, too, had been deeply influenced by Frege's views, and many parts of the *Tractatus* are devoted to them. Finally, we mention Rudolf Carnap, who had attended Frege's lectures at Jena – he describes how Frege lectured into the blackboard so that the handful of students in the room could barely hear him – and whose book *Meaning and Necessity* resuscitated interest in Frege and formal semantics.

Frege retired from Jena in 1918. He had became increasingly involved with right-wing political organizations toward the latter part of his life, and the journal he kept in spring 1924[9] reveals a side of him that is not very appealing.

2

Function and Argument

2.1 Introduction

Begriffsschrift was, as the subtitle announced, *a formula language of pure thought modeled upon the language of arithmetic*. Frege borrowed the notation for functions from arithmetic, and enlarged the realm of applicability of a function beyond the domain of numbers. Then, supplanting the subject/predicate division, which was characteristic of previous logical systems, by a function/argument division, he created a logical notation, a *Begriffsschrift* – literally, *Concept Writing* – which would serve to represent thoughts about any objects whatsoever. Like the language of arithmetic, his *Begriffsschrift* represented thoughts so that the inferential connections between them were molded in the representations themselves. The project was enormously successful. Not only did Frege create modern quantificational logic, but he also provided the theoretical framework for many subsequent philosophical developments in logic as well as in speculative philosophy. As Dummett (1981*a*) correctly remarked, Frege's work shifted the central focus of philosophy from the epistemological issues raised by Descartes back to the metaphysical and ontological issues that were salient after Aristotle.

The function/argument analysis Frege (1879) presented was, however, flawed. There was a significant confusion in his operating semantic notion of the *content* [*Inhalt*] of a sentence. Frege came to recognize that repairs were needed, and after much hard philosophical work, the theory with which we are now familiar emerged in the early 1890s. It was announced first in Frege (1891), and then elaborated upon in Frege (1892*c*) and Frege (1892*a*). We will present Frege's mature function/argument analysis, and later on, when we discuss the sense/reference distinction, explain some of the changes he had to make in his earlier theory. There is no place we are aware of where a reader can find the function/argument structure spelled out in any detail, so we have taken the liberty of presenting it here. The reader for whom this material is too elementary can simply leap to the next chapter.

2.2 What Is a Function?

The modern notion of a function goes like this. For any nonempty sets, S and S' (not necessarily distinct), a function f from S to S' correlates elements of S (*the domain of* f) with elements of S' (*the range of* f). If $x \in S$, then $f(x) \in S'$ and $f(x)$ is *the value* of the function f for *the argument* x. We are justified in speaking of *the* value of the function for a given argument because of the following fundamental property of functions:

Principle 2.2.1 (Fundamental Property of Functions) *For any x, y in the domain of f, if $x = y$, then $f(x) = f(y)$.*

Hence, f associates each element of S with but a single element of S'.[1]

A function is a special type of a *relation*, one that associates each element of the domain with a unique element of the range. Of course, a given element in the domain might be associated with more than one element of the range. In that case, however, the association is a relation that is not a function. *Being the brother of*, for example, is a relation that is not a function: it associates an individual with his brother(s). *Being the square root of* is an arithmetic relation that is not a function: although we speak of *the* square root of 4, we speak misleadingly, for there are two square roots of 4, +2 and −2.

Set-theoretically, relations and functions are conceived of as n-tuples of elements. A two-place relation, for example, will be a subset of the Cartesian Product $S \times S'$, that is, the set of ordered pairs $< x, y >$, with $x \in S$ and $y \in S'$ (S and S' not necessarily distinct). Principle 2.2.1 tells us that if $y = z$ whenever $< x, y >$ and $< x, z >$ are both in the relation, then that relation is also a function.

Before continuing, a word of caution is in order. Frege did not identify a function with a set of ordered pairs. The set of ordered pairs corresponds rather to what he called the *Werthverlauf* – the value range or course of values – of the function. We will see in Chapter 5 that Frege maintained a fundamental ontological division between objects [*Gegenstände*] on the one hand and functions on the other (corresponding roughly – very roughly – to the traditional distinction between objects and properties). The former, among which he counted *Werthverläufe*, are complete, self-subsistent entities; the latter are not self-subsistent, but, continuing Frege's metaphors, are unsaturated and stand in need of completion. However, for Frege, functions are the same if they yield the same values for the same arguments. Since this is in accord with the extensional view we are used to, we can rely on our set-theoretic intuitions as heuristic whenever ontological considerations fade into the background.

Here are some examples of arithmetic functions. The square function $f(x) = x^2$ is a singulary function, that is, a function of one argument. It maps integers into integers, associating each integer with its square: it maps 1 into 1, 2 into 4, 3 into 9, and so on. Addition, $f(x,y) = x + y$, is a *binary* function. It maps a pair of integers into integers: it maps the pair $< 1, 1 >$ into 2, it maps the pair $< 2, 3 >$ into 5, and so on.

In speaking as we have of functions, we have said very little about how the association is to be set up, or how the function is to be evaluated for a given argument. The set-theoretic perspective bypasses this important feature of the algebraic character of functions, which is crucial to our intuitive understanding of the notion. For example, when we consider the square function, expressed algebraically as $f(x) = x^2$, we think of the function as a way of getting from one number (the argument) to another (the value). It is the well-known mathematical procedure associated with the algebraic formula that gives the sense that the association between the domain and range is *orderly*. It is actually a rather large leap to suppose that a set of ordered pairs satisfying Principle 2.2.1 is a function, even when no procedure is available for associating the elements of the domain with the elements of the range. We are not sure how Frege would stand on this issue. We are inclined to believe that without an algebraic formula he would not be so quick to accept the existence of a function, because he posited sets *only* as the extensions of concepts – without the concept to identify the elements of the set, one could not otherwise assume the existence of such a set. But we cannot be sure of this.

2.3 Function and Argument

We now rehearse the analysis of the function/argument notation in mathematics, drawing mainly from Frege (1891). The linear function

$$f(x) = (2 \cdot x) + 1 \tag{2.1}$$

maps integers into integers. For the arguments 1, 2, and 3, the function yields the values 3, 5, and 7, respectively. Frege observes that the arithmetic equation

$$\text{`}3 = (2 \cdot 1) + 1\text{'} \tag{2.2}$$

is an identity, in fact, a true identity.[2] (2.2) says that the number 3 is identical with the number which is obtained by adding 1 to the result of multiplying 2 by 1. Since '$(2 \cdot 1) + 1$' flanks the identity sign in (2.2), it serves as a name: it designates the number that is obtained by adding 1 to the result of multiplying 2 by 1, namely, the number 3.

Unlike the numeral '3', however, which is a *simple* referring expression, '$(2 \cdot 1) + 1$' is a *complex* referring expression: it contains numerals as proper parts along with the symbols for addition and multiplication. The complex expression '$(2 \cdot 1) + 1$' was constructed by replacing the variable 'x' in the right-hand side of the equation in (2.2) by the numeral '1'. Now,

$$\text{`}(2 \cdot x) + 1\text{'} \tag{2.3}$$

does not stand for a number, and it especially does not stand for a variable or indefinite number as some of Frege's contemporaries were inclined to suppose. To prevent just such an error, Frege preferred to leave the variable 'x' out entirely and enclose the remaining blank space in parentheses, so that (2.3) would become

$$\text{`}(2 \cdot (\ )) + 1\text{'}, \tag{2.4}$$

an evidently incomplete expression. Though preferred, this notation is deficient when we have a function of more than one argument because we lose the difference in the variables that shows when we must insert the same numeral and when we need not. Frege eventually compromised by using the lower case Greek η and ζ instead of the blank spaces, writing (2.4) as

$$\text{`}(2 \cdot \eta) + 1\text{'}. \tag{2.5}$$

Again, (2.5) does not stand for a number, but rather, Frege suggested, for the linear function we started out with, namely,

$$(2 \cdot \eta) + 1. \tag{2.6}$$

Frege's suggestion captures the notation beautifully. For, inserting the numerals

$$\text{`}1\text{'}, \text{`}2\text{'}, \text{`}3\text{'} \tag{2.7}$$