1

Introduction

The topic of this book is the application of mathematics to physical problems. Mathematics and physics are often taught separately. Despite the fact that education in physics relies on mathematics, it turns out that students consider mathematics to be disjoint from physics. Although this point of view may strictly be correct, it reflects an erroneous opinion when it concerns an education in the sciences. The reason for this is that mathematics is the *only* language at our disposal for quantifying physical processes. One cannot learn a language by just studying a textbook. In order to truly learn how to use a language one has to go abroad and start using that language. By the same token one cannot learn how to use mathematics in the physical sciences by just studying textbooks or attending lectures; the only way to achieve this is to venture into the unknown and apply mathematics to physical problems.

It is the goal of this book to do exactly that; problems are presented in order to apply mathematical techniques and knowledge to physical concepts. These examples are not presented as well-developed theory. Instead, they are presented as a number of problems that elucidate the issues that are at stake. In this sense this book offers a guided tour: material for learning is presented but true learning will only take place by active exploration. In this process, the interplay of mathematics and physics is essential; mathematics is the natural language for physics while physical insight allows for a better understanding of the mathematics that is presented.

How can you use this book most efficiently?

Since this book is written as a set of problems you may frequently want to consult other material as well to refresh or deepen your understanding of material. In many places we refer to the book of Boas [19]. In addition, the books of Butkov [24], Riley *et al.* [87] and Arfken [5] on mathematical physics are excellent.

In addition to books, colleagues in either the same field or other fields can be a great source of knowledge and understanding. Therefore, do not hesitate to work together with others on these problems if you are in the fortunate position to do so. This may not only make the work more enjoyable, it may also help you in getting "unstuck" at difficult moments and the different viewpoints of others may help to deepen yours.

For who is this book written?

This book is set up with the goal of obtaining a good working knowledge of mathematical physics that is needed for students in physics or geophysics. A certain basic knowledge of calculus and linear algebra is required to digest the material presented here. For this reason, this book is meant for upper-level undergraduate students or lower-level graduate students, depending on the background and skill of the student. In addition, teachers can use this book as a source of examples and illustrations to enrich their courses.

This book is evolving

This book will be improved regularly by adding new material, correcting errors and making the text clearer. The feedback of both teachers and students who use this material is vital in improving this text, please send your remarks to:

Roel Snieder

Dept of Geophysics

Colorado School of Mines
Golden CO 80401
USA

telephone: +1-303-273.3456
fax: +1-303-273.3478
email: rsnieder@mines.edu

Errata can be found at the following website: www.mines.edu/~rsnieder/Errata.html.

2

Dimensional analysis

The material of this chapter is usually not covered in a book on mathematics. The field of mathematics deals with numbers and numerical relationships. It does not matter what these numbers are; they may account for physical properties of a system, but they may equally well be numbers that are not related to anything physical. Consider the expression $g = df/dt$. From a mathematical point of view these functions can be anything, as long as g is the derivative of f. The situation is different in physics. When $f(t)$ is the position of a particle, and t denotes time, then $g(t)$ is a velocity. This relation fixes the physical dimension of $g(t)$. In mathematical physics, the physical dimension of variables imposes constraints on the relation between these variables. In this chapter we explore these constraints. In Section 2.2 we show that this provides a powerful technique for spotting errors in equations. In the remainder of this chapter we show how the physical dimensions of the variables that govern a problem can be used to find physical laws. Surprisingly, while most engineers learn about dimensional analysis, this topic is not covered explicitly in many science curricula.

2.1 Two rules for physical dimensions

In physics every physical parameter is associated with a physical dimension. The value of each parameter is measured with a certain physical unit. For example, when I measure how long a table is, the result of this measurement has dimension "length". This length is measured in a certain unit, that may be meters, inches, furlongs, or whatever length unit I prefer to use. The result of this measurement can be written as

$$l = 3 \text{ m}. \tag{2.1}$$

The variable l has the physical dimension of length, in this chapter we write this as

$$l \sim [L]. \tag{2.2}$$

The square brackets are used in this chapter to indicate a physical dimension. The capital letter L denotes length, T denotes time, and M denotes mass. Other physical dimensions include electric charge and temperature. When dealing with physical dimensions two rules are useful. The first rule is:

Rule 1 When two variables are added, subtracted, or set equal to each other, they must have the same physical dimension.

In order to see the logic of this rule we consider the following example. Suppose we have an object with a length of 1 meter and a time interval of one second. This means that

$$\begin{aligned} l &= 1 \text{ m}, \\ t &= 1 \text{ s}. \end{aligned} \tag{2.3}$$

Since both variables have the same numerical value, we might be tempted to declare that

$$l = t. \tag{2.4}$$

It is, however, important to realize that the physical units that we use are arbitrary. Suppose, for example, that we had measured the length in feet rather than meters. In that case the measurements (2.3) would be given by

$$\begin{aligned} l &= 3 \text{ ft}, \\ t &= 1 \text{ s}. \end{aligned} \tag{2.5}$$

Now the numerical value of the same length measurement is different! Since the choice of the physical units is arbitrary, we can scale the relation between variables of different physical dimensions in an arbitrary way. For this reason these variables cannot be equal to each other. This implies that they cannot be added or subtracted either.

The first rule implies the following rule.

Rule 2 Mathematical functions can act on dimensionless numbers only.

To see this, let us consider as an example the function $f(\xi) = e^{\xi}$. Using a Taylor expansion, this function can be written as:

$$f(\xi) = 1 + \xi + \frac{1}{2}\xi^2 + \cdots \tag{2.6}$$

According to rule 1 the different terms in this expression must have the same physical dimension. The first term (the number 1) is dimensionless, hence all the other terms in the series must be dimensionless. This means that ξ must be a dimensionless number as well. This argument can be used for any function $f(\xi)$ whose Taylor expansion contains different powers of ξ. Note that the argument would not hold for a function such as $f(\xi) = \xi^2$ that contains only one power of ξ. To please the purists, rule 2 could easily be reformulated to exclude these special cases.

These rules have several applications in mathematical physics. Suppose we want to find the physical dimension of a force, as expressed in the basic dimensions mass, length, and time. The only thing we need to do is take one equation that contains a force. In this case Newton's law $F = ma$ comes to mind. The mass m has physical dimension $[M]$, while the acceleration has dimension $[L/T^2]$. Rule 1 implies that force has the physical dimension $[ML/T^2]$.

Problem a The force F in a linear spring is related to the extension x of the spring by the relation $F = -kx$. Show that the spring constant k has dimension $[M/T^2]$.

Problem b The angular momentum $\mathbf{L}$ of a particle with momentum $\mathbf{p}$ at position $\mathbf{r}$ is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{2.7}$$

where $\times$ denotes the cross-product of two vectors. Show that angular momentum has the dimension $[ML^2/T]$.

Problem c A plane wave is given by the expression

$$u(\mathbf{r}, t) = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{2.8}$$

where $\mathbf{r}$ is the position vector and t denotes time. Show that $\mathbf{k} \sim [L^{-1}]$ and $\omega \sim [T^{-1}]$.

In quantum mechanics the behavior of a particle is characterized by a wave equation, that is called the Schrödinger equation. In one space dimension this equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi, \tag{2.9}$$

where x denotes the position, t denotes the time, m the mass of the particle, and $V(x)$ the potential energy of the particle. At this point it is not clear what the wave function $\psi(x, t)$ is, and how this equation should be interpreted. The meaning of the symbol $\hbar$ is not yet defined. We can, however, determine the physical dimension of $\hbar$ without knowing the meaning of this variable.

Problem d Compare the physical dimensions of the left-hand side of (2.9) with the first term on the right-hand side and show that the variable $\hbar$ has the physical dimension angular momentum. You can use problem b in showing this.

2.2 A trick for finding mistakes

The requirement that all terms in an equation have the same physical dimension is an important tool for spotting mistakes. Cipra [26] gives many useful tips for spotting errors in his delightful book "Misteakes [sic] . . . and how to find them before the teacher does." As an example of using dimensional analysis for spotting mistakes, we consider the erroneous equation

$$E = mc^3, \tag{2.10}$$

where E denotes energy, m denotes mass, and c is the speed of light. Let us first find the physical dimension of energy. The work done by a force $\mathbf{F}$ over a displacement $d\mathbf{r}$ is given by $dE = \mathbf{F} \cdot d\mathbf{r}$. We showed in Section 2.1 that force has the dimension $[ML/T^2]$. This means that energy has the dimension $[ML^2/T^2]$. The speed of light in the right-hand side of expression (2.10) has dimension $[L/T]$, which means that the right-hand side has physical dimension $[ML^3/T^3]$. This is not an energy, which has dimension $[ML^2/T^2]$. Therefore expression (2.10) is wrong.

Problem a Now that we have determined that expression (2.10) is incorrect we can use the requirement that the dimensions of the different terms must match to guess how to set it right. Show that the right-hand side must be divided by a velocity to match the dimensions.

It is not clear that the right-hand side must be divided by the speed of light to give the correct expression $E = mc^2$. Dimensional analysis tells us only that it must be divided by something with the dimension of velocity. For all we know, it could be the speed at which the average snail moves.

Problem b Is the following equation dimensionally correct?

$$(\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p. \tag{2.11}$$

In this expression $\mathbf{v}$ is the velocity of fluid flow, p is the pressure, and ∇ is the gradient vector (which essentially is a derivative with respect to the space coordinates). You can use that pressure has the dimension is force per unit area.

Problem c Answer the same question for the expression that relates the particle velocity v to the pressure p in an acoustic medium:

$$v = \frac{p}{\rho c} \tag{2.12}$$

Here ρ is the mass density and c is velocity of propagation of acoustic waves.

Problem d In quantum mechanics, the energy E of the harmonic oscillator is given by

$$E_n = \hbar\omega^2 (n + 1/2) \, , \tag{2.13}$$

where ω is a frequency, n is a dimensionless integer, and $\hbar$ is Planck's constant divided by 2π as introduced in problem d of the previous section. Verify if this expression is dimensionally correct.

In general it is a good idea to carry out a dimensional analysis while working in mathematical physics because this may help in finding the mistakes that we all make while doing derivations. It takes a little while to become familiar with the dimensions of properties that are used most often, but this is an investment that pays off in the long run.

2.3 Buckingham pi theorem

In this section we introduce the Buckingham pi theorem. This theorem can be used to find the relation between physical parameters based on dimensional arguments. As an example, let us consider a ball shown in Figure 2.1 with mass m that is dropped from a height h. We want to find the velocity with which it strikes the ground. The potential energy of the ball before it is dropped is mgh, where g is the acceleration of gravity. This energy is converted into kinetic energy $\frac{1}{2}mv^2$ as it strikes the ground. Equating these quantities and solving for the velocity gives:

$$v = \sqrt{2gh}. \tag{2.14}$$

Now let us suppose we did not know about classical mechanics. In that case, dimensional analysis could be used to guess relation (2.14). We know that the velocity is some function of the acceleration of gravity, the initial height, and the mass of the particle: $v = f(g, h, m)$. The physical dimensions of these properties

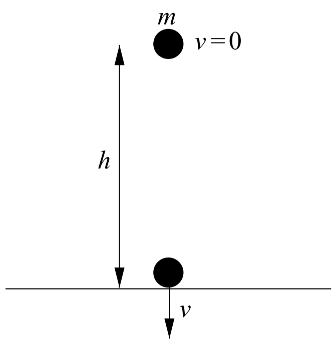

Fig. 2.1 Definition of the variables for a falling ball.

are given by

$$v \sim [L/T], \quad g \sim [L/T^2], \quad h \sim [L], \quad m \sim [M]. \tag{2.15}$$

Let us consider the dimension mass first. The dimension mass enters only the variable m. We cannot combine the variable m with the parameters g and h in any way to arrive at a quantity that is independent of mass. Therefore, the velocity does not depend on the mass of the particle. Next we consider the dimension time. The velocity depends on time as $\left[T^{-1}\right]$, the acceleration of gravity as $\left[T^{-2}\right]$, and h is independent of time. This means that we can match the dimension time only when

$$v \sim \sqrt{g}. \tag{2.16}$$

In this expression the left-hand side depends of the length as $[L]$, while the right-hand side varies with length as $[L^{1/2}]$. We have, however, not used the height h yet. The dimension length can be made to match if we multiply the right-hand side with $h^{1/2}$. This means that the only combination of g and h that gives a velocity is given by

$$v \sim \sqrt{gh}. \tag{2.17}$$

This result agrees with expression (2.14), which was derived using classical mechanics. Note that in order to arrive at expression (2.17) we used only dimensional arguments, and did not need to have any knowledge from classical mechanics other than that the velocity depends only on g and h. The dimensional analysis that led to expression (2.17), however, does not tell us what is the proportionality constant in that expression. The reason is that a proportionality constant is dimensionless, and can therefore not be found by dimensional analysis.

The treatment given here may appear to be cumbersome. This analysis, however, can be carried out in a systematic fashion using the *Buckingham pi theorem* [23] which states the following:

Buckingham pi theorem If a problem contains N variables that depend on P physical dimensions, then there are $N - P$ dimensionless numbers that describe the physics of the problem.

The original paper of Buckingham is very clear, but as we will see at the end of this section, this theorem is not fool-proof. Let us first apply the theorem to the problem of the falling ball. We have four variables: v, g, h, and m, so that $N = 4$. These variables depend on the physical dimensions $[M]$, $[L]$, and $[T]$, hence $P = 3$. According to the Buckingham pi theorem, $N - P = 1$ dimensionless number characterizes the problem. We want to express the velocity in the other parameters; hence we seek a dimensionless number of the form

$$v g^{\alpha} h^{\beta} m^{\gamma} \sim [1], \tag{2.18}$$

where the notation in the right-hand side means that it is dimensionless. Let us seek the exponents α, β, and γ that make the left-hand side dimensionless. Inserting the dimensions of the different variables then gives the following dimensions

$$\left[\frac{L}{T}\right]\left[\frac{L^{\alpha}}{T^{2\alpha}}\right]\left[L^{\beta}\right]\left[M^{\gamma}\right] \sim [1]\,. \tag{2.19}$$

The left-hand side depends on length as $[L^{1+\alpha+\beta}]$. The left-hand side can only be independent of length when the exponent is equal to zero. Applying the same reasoning to each of the dimensions length, time, and mass, then gives

$$\begin{array}{ll} \text{dimension } [L]: & 1 + \alpha + \beta = 0, \\ \text{dimension } [T]: & -1 - 2\alpha = 0, \\ \text{dimension } [M]: & \gamma = 0. \end{array} \tag{2.20}$$

This constitutes a system of three equations with three unknowns.

Problem a Show that the solution of this system is given by

$$\alpha = \beta = -\frac{1}{2}, \qquad \gamma = 0. \tag{2.21}$$

Inserting these values into expression (2.18) shows that the combination $v g^{-1/2} h^{-1/2}$ is dimensionless. This implies that

$$v = C\sqrt{gh}, \tag{2.22}$$

where C is the one dimensionless number in the problem as dictated by the Buckingham pi theorem.

The approach taken here is systematic. In his original paper [23], Buckingham applied this treatment to a number of problems: the thrust provided by the screw of a ship, the energy density of the electromagnetic field, the relation between the mass and radius of the electron, the radiation of an accelerated electron, and heat conduction.

There is, however, a catch that we introduce with an example. When air (or water) has a stably stratified mass–density structure, it can support oscillations where the restoring force is determined by the density gradient in the air. These oscillations occur with the *Brunt-Väisälä* frequency ω_B given by [50, 82]:

$$\omega_B = \sqrt{\frac{g}{\theta}\frac{d\theta}{dz}}. \tag{2.23}$$

In this expression, g is the acceleration of gravity, z is height, and θ is potential temperature (a measure of the thermal structure of the atmosphere).

Problem b Verify that this expression is dimensionally correct.

Problem c Check that this expression is also dimensionally correct when θ is replaced by the air pressure p, or the mass density ρ.

The result of problem c indicates that the potential temperature θ can be replaced by any physical parameter, and expression (2.23) is still dimensionally correct. This means that a dimensional analysis alone can never be used to prove that θ should be the potential temperature. In order to show this we need to know more of the physics of the problem.

Another limitation of the Buckingham pi theorem as formulated in its original form is that the theorem assumes that physical parameters need to be multiplied or divided to form dimensionless numbers; see equation (3) of reference [23]. The derivative of one variable with respect to another, however, has the same dimension as the ratio of these variables. Consider for example a problem where dimensional analysis shows that the variable of interest depends on the ratio of the acceleration of gravity and the height: g/h. The derivative of g with height dg/dz has the same physical dimension as g/h. Therefore, a dimensional analysis alone cannot completely describe the physics of the problem. Nevertheless, as we will see in the following section, it may provide valuable insights.