Elementary Euclidean Geometry An Introduction

This is a genuine introduction to the geometry of lines and conics in the Euclidean plane. Lines and circles provide the starting point, with the classical invariants of general conics introduced at an early stage, yielding a broad subdivision into types, a prelude to the congruence classification. A recurring theme is the way in which lines intersect conics. From single lines one proceeds to parallel pencils, leading to midpoint loci, axes and asymptotic directions. Likewise, intersections with general pencils of lines lead to the central concepts of tangent, normal, pole and polar.

The treatment is example-based and self-contained, assuming only a basic grounding in linear algebra. With numerous illustrations and several hundred worked examples and exercises, this book is ideal for use with undergraduate courses in mathematics, or for postgraduates in engineering and the physical sciences.

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C. G. GIBSON



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Preface

It is worth saying something about the background to this book, since it is linked to a sea change in the teaching of university mathematics, namely the renaissance in undergraduate geometry, following a postwar decline. There is little doubt that the enormous progress made in studying non-linear phenomena by geometric methods has rekindled interest in the subject. However, that is not the only reason for seeking change, as I pointed out in the preface to *Elementary Geometry of Algebraic Curves*:

'For some time I have felt there is a good case for raising the profile of undergraduate geometry. The case can be argued on *academic* grounds alone. Geometry represents a way of thinking within mathematics, quite distinct from algebra and analysis, and so offers a fresh perspective on the subject. It can also be argued on purely *practical* grounds. My experience is that there is a measure of concern in various practical disciplines where geometry plays a substantial role (engineering science for instance) that their students no longer receive a basic geometric training. And thirdly, it can be argued on *psychological* grounds. Few would deny that substantial areas of mathematics fail to excite student interest: yet there are many students attracted to geometry by its sheer visual content.'

Background

A good starting point in developing undergraduate geometry is to focus on plane curves. They comprise a rich area, of historical significance and increasing relevance in the physical and engineering sciences. That raises a practical consideration, namely that there is a dearth of suitable course texts: some are out of date, whilst others are written at too high a level, or contain too much material. Cambridge University Press 978-0-521-83448-3 - Elementary Euclidean Geometry: An Introduction C. G. Gibson Frontmatter More information

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I felt it was time to improve the situation, bearing in mind the importance of foundational mathematical training, where the primary objective is to enable students to gain fluency in the basics. (Those who wish to develop their interests will be warmly welcomed at the postgraduate level.) Over my career, one of the healthier developments in the teaching of university mathematics is the widespread adoption of clean, careful treatments of foundational material. For instance linear algebra, group theory, general abstract algebra, introductory calculus and real analysis are now widely taught on this pattern, supported by excellent texts. Such courses fit the contemporary mould of good mathematics education, by exhibiting internal coherence, an intrinsic approach, and standards of proof appropriate to the subject. I wanted to see geometry regain its place in the mathematics curriculum, within this broad pattern.

The Elementary Geometry Trilogy

It was against this background that I wrote two companion texts¹ presenting elementary accounts of complementary viewpoints, to wit the *algebraic* viewpoint (where curves are defined by the vanishing of a polynomial in two variables) and the *differentiable* viewpoint (where curves are parametrized by a single real variable). I have been encouraged by the reactions of the mathematical community, which has welcomed these contributions to undergraduate geometry.

Both texts were intended primarily for second year students, with later material aimed at third years. However, neither addresses the question of introducing university students to geometry *for the first time*. I emphasize this for good reason, namely that geometry has largely disappeared from school mathematics. In my experience, few students acquire more than an imperfect knowledge of lines and circles before embarking on their degree studies.

I think the way forward is to offer foundational geometry courses which properly expose the body of knowledge common to both viewpoints, the basic geometry of lines and conics in the Euclidean plane. The geometry of conics is important in its own right. Conics are of considerable historical significance, largely because they arise naturally in numerous areas of the physical and engineering sciences, such as astronomy, electronics, optics, acoustics, kinematics, dynamics and architecture. Quite apart from their physical importance, conics are quite fundamental objects in mathematics itself, playing crucial roles in understanding general plane curves.

¹ Elementary Geometry of Algebraic Curves and Elementary Geometry of Differentiable Curves, published by Cambridge University Press, and henceforth referred to as *EGAC* and *EGDC* respectively. The present text will be designated as *EEG*.

Preface

In this respect *EEG* should be of solid practical value to students and teachers alike. On such a basis, students can develop their geometry with a degree of confidence, and a useful portfolio of down-to-earth examples. For teachers, *EEG* provides a source of carefully worked out material from which to make a selection appropriate to their objectives. Such a selection will depend on several factors, such as the attainment level of the students, the teaching time available, and the intended integration with other courses.

I make no apology for the fact that some sections overlap the material of EGAC and EGDC. On the contrary, I saw close integration as a positive advantage. This book is a convenient stepping stone to those texts, taking one further down the geometry road, and bringing more advanced treatments within reach. In this way *EEG* can be viewed as the base of a trilogy, sharing a common format. In particular, the book is unashamedly example based. The material is separated into short chapters, each revolving around a single idea. That is done for good pedagogical reasons. First, students find mathematics easier to digest when it is split into a bite-sized chunks: the overall structure becomes clearer, and the end of each chapter provides a welcome respite from the mental effort demanded by the subject. Second, by pigeon-holing the material in this way the lecturer gains flexibility in choosing course material, without damaging the overall integrity. On a smaller scale, the same philosophy is pursued within individual chapters. Each chapter is divided into a number of sections, and in turn each section is punctuated by a series of 'examples', culminating in 'exercises' designed to illustrate the material, and to give the reader plenty of opportunity to master computational techniques and gain confidence.

Axioms for Writing

The material is designed to be accessible to those with minimal mathematical preparation. Basic linear algebra is the one area where some familiarity is assumed: the material of a single semester course should suffice. And it would be an advantage for the reader to feel comfortable with the concept of an equivalence relation.

One of my guiding axioms was that the content should provide the reader with a secure foundation for further study. Though elementary, it is coherent mathematics, not just a mishmash of calculations posing as geometry. There are new ways of viewing old things, concepts to be absorbed, results to contemplate, proofs to be understood, and computational techniques to master, all of which further the student's overall mathematical development. In this respect I feel it is important for the student to recognise that although geometric intuition points one in the right direction, it is no substitute for formal proof. xiv

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To be consistent with that philosophy, it is necessary to provide intrinsic definitions and argue coherently from them.

There is something to be said for ring-fencing the content of a foundational course from the outset. In the present context, I felt there was a good case for restricting the geometry entirely to the real Euclidean plane. For instance, even with that restriction there is more than enough material from which to choose. Also, at the foundational level it may be unwise to develop too many concepts. Thus complex conics are probably best left till students feel comfortable with the mechanics of handling complex numbers. Likewise, my experience suggests it is sensible to leave the projective plane till a little later in life.

The Development

One has to maintain a careful balance between theory and practice. For instance, the initial discussion of lines emphasizes the difference between a linear function on the plane and its zero set. To the student that may seem unduly pedantic, but failure to make the distinction introduces a potential source of confusion. On the other hand, since lines are quite fundamental to the development, a whole section is devoted to the practicalities of handling them efficiently. The Euclidean structure on the plane may well be familiar from a linear algebra course: nevertheless, there is a self–contained treatment, leading to the formula for the distance from a point to a line which underlies the focal constructions of conics.

Circles provide the first examples of general conics, and of the fact that a conic may not be determined by its zero set. However, we follow the pattern for lines by showing that the zero sets of *real* circles do determine the equation, a result extended (in the final chapter) to general conics with infinite zero sets. From circles it is but a short step to general conics. The classical invariants are introduced at an early stage, yielding a first broad subdivision into types, a prelude to the later congruence classification. Despite their uninteresting geometry, degenerate conics do arise naturally in families of conics as transitional types: and for that reason, a chapter is devoted to them. Likewise, a chapter is reserved for centres, since they provide basic geometric distinctions exploited in the congruence classification.

A recurring theme in the development is the way in which lines intersect conics. From single lines we progress to parallel pencils, leading to the classical midpoint locus, and the concepts of axis and asymptotic direction. In the same vein we study pencils of lines through a point on a conic, leading to the central geometric concepts of tangent and normal. Finally, the question of how Preface

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a general pencil of lines interesects a conic gives rise to the classical concepts of pole and polar, and the interesting idea of the orthoptic locus.

This text has two distinctive features. The first is that despite its intrinsic importance to the metric geometry, the classical focal construction appears later in the development than is usual. That is quite deliberate. One reason is that it aids clarity of thought. But there is also a technical reason. I wanted a method for finding foci and directrices *independent of the congruence classification*. That not only enables the student to handle a wider range of examples, but also clarifies the uniqueness question for focal constructions, a surprising omission in most texts. Another distinctive feature is that the congruence classification is left till the end. Again, that is quite deliberate. To my way of thinking, the geometry is more interesting than the listing process, so deserves to be developed first. Also, the congruence classification is a natural resting point in the student's geometric progression. Looking back, it lends cohesion to the range of examples met in the text: and looking forward, it raises fundamental questions which are better left to final year courses.

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